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MEAN VARIANCE OF FRACTIONAL STOCHASTIC MODEL AND LOGARITHM UTILITY OPTIMIZATION OF A PENSION FUND WITH TAX AND TRANSACTION COST

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ABSTRACT

This work looked at the mean-variance of fractional continuous time stochastic model for the dynamics of a pension fund with tax and transaction cost, where the effect of tax and transaction cost charging makes on the expected logarithmic utility of the pensioner was established. The associated H-J-B equation in the optimization problem is obtained using lto's lemma. An explicit solution to the pensioners' problems was derived under stated condition.

KEYWORDS

Mean-variance portfolio selection; Brownian motion; H-J-B Equation; Pensioner logarithmic utility, mode of taxation, transaction cost ;consumption ; reinsurance

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Introduction:

The mean-variance portfolio selection problem has provided a fundamental basis for portfolio construction and has stimulated hundreds of extensions and applications. Li and Ng [5] and Bauerle [10] are the first to extend the original static mean-variance model to multi-period and continuous-time cases respectively. Since then, various kinds of problems under mean-variance criterion have been investigated analytically. For instances, Wu and Li [6] and Xie *at al* [14] considered an asset and liability management problem. Olunkwa *et-al*[5] Considered two type of volatility, including a fast –moving one and a slowly-moving one by using the stochastic dynamic programming principle and Hamilton-Jacobi-Bellman equation approach.

Merton [11] proposed the stochastic control approach to study the investment problem for the first time and hence marks the beginning of utility maximization which has been an important issue in optimal investment problems in mathematical finance and has drawn great attention in recent years. Pliska [13], Karatzas [7] adapted the martingale approach to investment problems of utility maximization. Zhang [2] investigated the utility maximization problem in an incomplete market using the martingale approach. Investment and reinsurance are two important ways for insurers to balance their profit and risk. Recently, the problem of optimal investment and/or reinsurance for insurers has been extensively studied in the literature. For example, in the framework of utility maximization, Bai and Guo [8], Cao and Wan [17], Irgens and Paulsen [4], and Liang et al. [18] discuss optimal proportional reinsurance-investment problems and Asmussen et al. [12], Gu et al. [1], and Zhang et al. [16] consider optimal excess-of-loss reinsurance-investment problems; in the framework of mean-variance, Bai and Zhang [9], Bauerle [10], Li et al. [19], Zeng et al. [20], and Zeng and Li [21] study optimal investment and/or reinsurance problem. S. A. Ihedioha1 and C. Olunkwa [22] looked at the impact of mode of taxation and transaction costs on the insurance company.

In this paper we discussed the mean variance of fractional stochastic model and logarithm utility optimization of a pension fund with tax and transaction cost. A sensitive analysis was also done.

The Model

In our model of the pension plan the assets of the pensioners are invested in a stock market index. The value of the index is the market capitalization weighted average of its components' sock prices. We assume that the market is made up of risk –free asset (saving) and risky asset (stock).

Let the price of the risk free asset and risky asset be denoted by $B(t) and\, S(t)$ respectively and equation governing the dynamics of the risk free asset is given by

$$dB_t(t) = rB_t(t)dt \quad (1)$$

Let (Ω, F, P) be a complete probability space where Ω a real space is and P is a probability measure. In a define Contribution (DC) fund system, members remit certain proportion of their salary to the pension account every month. (The total contribution (both of employer and employee) is a constant fractionζ of the salary.Hence we assume that the number of contributors are constant P. Grimberg and Z. Schuss [23] and the contribution rate is modeled as

$$c_i(t) = \zeta\, s_i(t) \quad (2)$$

We are concerned with investment behavior in the presence of a stochastic cash flow or risk process denoted by $Z_n(t): t \geq 0$ which describe a fractional linear Brownian motion

$$dZ_n(t) = \psi(t)Z_n(t)dt + \phi(t)Z_n(t)dw_H^1(t) \qquad (3)$$

where $\psi(t)$ is the appreciation rate given as $\psi(t) = r(t) + \xi(t)\sigma(t)$ and $\phi(t)$ is the volatility of the risky assets.

We incorporate the fractional stochastic differential equation for risky asset to form the surplus in the pension fund. For every $0 \leq J \leq n$ the contribution in naira amount at time t_{ij} is given by $\alpha_i c_i(t_{ij})$,We have that the members first salary contribution is α_i,where the appreciation of this amount is compounded from t_i through t_{in} and is given by

$$\frac{Z_n(t_{in})}{Z_n(t_{ij})} \qquad (5a)$$

The portion of the pensions total amount attributed to $j-th$ is give by

$$\alpha_i c_i(t_{ij})\frac{Z_n(t_{in})}{Z_n(t_{ij})} \qquad (5b)$$

And the portion of the pension's surplus contribution is obtained from equation(5) by division by α_i.Therefore the surplus of the pension fund from time $t_{i0}(first\ salary) to\ time\ t(retairment)$ is given by

$$V_i(t) = \sum_{\tau \epsilon T_i} c_i(\tau)\frac{Z_n(t)}{Z_n(\tau)} = Z_n(t)\sum_{\tau \epsilon T_i}\frac{c_i(\tau)}{Z_n(\tau)} \qquad (6)$$

The continuous model for $V_i(t)$ is obtained by representing equation (6) as the Riemann sum

$$V_i(t) = \frac{Z_n(t)}{\Delta(t)}\sum_{j=1}^{n}\frac{c_i(j\Delta t)}{Z_n(j\Delta t)}\Delta t \qquad (7)$$

where $\Delta t = (t_{ij} - t_{ij-1})$is the constant time elapsed between consecutive salaries.Writing equation(7) in integral form

$$V_i(t) = Z_n(t)\left(\int_{t_{io}}^{t}\frac{c_i(u)}{Z_n(u)}\right) \qquad (8)$$

we obtained the fractional stochastic model for surplus of the pension fund $V_i(t)$ by differentiating equa8 and applying the chain rule.

$$dV_i(t) = dZ_n(t)\left(\int_{t_{io}}^{t}\frac{c_i(u)}{Z_n(u)}du\right) + Z_n(t)\left(\frac{c_i(t)}{Z_n(t)}dt\right) \quad (9)$$

substituting equation (3) into equation8 we obtain a fractional SDE of which is the surplus of the pension fund after investment. This is describe as the

$$dV_i(t) = [\psi(t)V_i(t) + \zeta s_i(t)]dt + \phi(t)V_i(t)dw_H^1(t)(t) \qquad (10a)$$

let the claim process of the pension fund be described as equation (10a).

assuming the premium rate is

$$c = \psi(t)V_i(t) + \zeta s_i(t) + \theta\left(\psi(t)V_i(t) + \zeta s_i(t)\right) \qquad (10b)$$

with security risk premium $\theta > 0$.The surplus process of the pension fund using equation 10a is given by

$$dR(t) = cdt - dV_i(t)$$

$$dR(t) = \theta\left(\psi(t)V_i(t) + \zeta s_i(t)\right)\mathrm{dt} + \phi(t)V_i(t)dw_H^1(t)$$

The pension fund company is permitted to purchase proportional reinsurance to reduce risk and pays premium at the rate $(1+\eta)\left(\psi(t)V_i(t) + \zeta s_i(t)\right)p(t)$continuously, where$\eta > \theta > 0$ is the securirty risk of the persioner and $p(t)$ is the proportion than is pensioned at time t,then the surplus of the pension fund is given by

$$dR(t) = (\theta + \eta p(t))\left(\psi(t)V_i(t) + \zeta s_i(t)\right)\mathrm{dt} + (1 - \mathrm{p(t)})\phi(t)V_i(t)dw_H^1(t)$$

The pension fund company invests her surplus in two assets, a risky and risk free asset.

here the short rate $r(t)$satisfies the differential equation

$$dr(t) = (\bar{a}(t) + b\xi(t))dt + bdw(t), \qquad r(0) = r_0,$$

Where $b > 0$ is a constant, the risk premum $\xi(r)$ is a deterministic and continuous function. Is a fractional Brownian motion and $\bar{a}(t) = \theta(t) - ar(t)$ is a stochastic process related to $r(t)$,here $\alpha > 0$ and $\theta(t)$ is deterministic and continuously differentiable function [20]

The risky asset is model by the fractional stochastic differential equation

$$dS_n(t) = S_n(t)[\mu dt + \beta dw_H^2(t)]$$

$w_H^2(t)$ is another fractional Brownian motion.$w_H^1(t)$ and $w_H^2(t)$ are allowed to correlate with correlation coefficient ρ that is

$$Cov\left(w^1(t), w^2(t)\right) = \rho t$$

The pension funds invest the risky assets as $\psi > r$

If $\pi(t)$ be the naira amount invested in the risky asset at time t and the remaining amount$[U(t) - \pi(t)]$ be the naira amount invested in the risk free asset, where $U(t)$ is the surplus process of the pension fund company (the company's total investment on both assets).

Let the rate at which tax are paid in the financial market be α, k rate of dividend income and λ the rate of transaction costs which consists of fees and stamp duties etc.

Two cases shall be considered:

1) The case where transaction costs and taxes are changed only on the risky investment .
2) The case where transaction costs and taxes and charged on the total investment of the pension fund

Assumptions:

The following assumption are made

a) The pension fund company makes intermediate consumption space which satisfies

$$\int_0^t |C(s)|ds < \infty, \forall\, t\epsilon[0,T]$$

b) Consumption is made through risk -free account only
c) Dividends are paid on investment in the risky asset only

Therefore, corresponding to the trading strategy $\pi(t)$ and intial capital U_0 ,the wealth process of the pension fund follows the dynamics of risk free asset given in equation 2.

CASE 1

When transaction cost and taxes are charged only on risky asset (individual pension)

$$dU^{\pi}(t) = \pi(t)\frac{dS_0(t)}{S_0(t)} + [U(t) - \pi(t)]\frac{dB(t)}{B(t)} + k\pi(t)dt - (\alpha+\lambda)\pi(t)dt - C(t)dt + dR(t)$$

Where C(t) is the consumption rate. Subtitling for $\frac{dS_0(t)}{S_0(t)}, \frac{dB(t)}{B(t)}$ $and\ dR(t)$.

$$dU^{\pi}(t) = \pi(t)[\mu dt + \beta dw_H^2(t)] + [U(t) - \pi(t)]rdt + k\pi(t)dt - (\alpha+\lambda)\pi(t)dt - C(t)dt + (\theta - \eta p(t))\big(\psi(t)V_i(t) - \zeta s_i(t)\big)\mathrm{dt} + (1 - \mathrm{p(t)})\phi(t)V_i(t)dw_H^1(t) \qquad (11)$$

This simplifies further to

$$dU^{\pi}(t) = \{(\mu + k) - (\alpha + \lambda + r)\pi(t) + ru + (\theta + \eta p(t))\big(\psi(t)V_i(t) - \zeta s_i(t)\big) - \mathrm{C(t)}\}dt + \pi(t)\beta dw_H^2(t) + (1 - \mathrm{p(t)})\phi(t)V_i(t)dw_H^1(t) \qquad (12)$$

CASE 2

When transaction cost and taxes are charged on the entire –investment

$$dU^{\pi}(t) = \pi(t)\frac{dS_0(t)}{S_0(t)} + [U(t) - \pi(t)]\frac{dB(t)}{B(t)} + k\pi(t)dt - (\alpha+\lambda)U(t)dt - C(t)dt + dR(t) \quad (13)$$

This simplifies to

$$dU^{\pi}(t) = [(\mu + k) - r]\pi(t) + [r - (\alpha+\lambda)]U(t) + \big(\theta - \eta p(t)\big)\phi(t)\psi(t)V_i(t) - \zeta s_i(t) - C(t)]dt + [\pi(t)\beta]^2 dw_H^2(t) + (1 - \mathrm{p(t)})\phi(t)V_i(t)dw_H^1(t) \qquad (14)$$

Since $w_H^1(t), w_H^2(t)$ are correlating standard Brownian motion with correlation coefficient σ and applying the rule.

$$dw_H^1(t).dw_H^2(t) = dw_H^2(t).dw_H^2(t) = dt, dw_H^1(t).dw_H^2(t) = dt.dt = 0 \qquad (15)$$

The quadratic variation of 12 and 14 is

$$\langle dU^{\pi}(t)\rangle = [\pi(t)\beta]^2 + \left\{\left[\left(1-\mathrm{p(t)}\right)\phi(t)V_i(t)\right]^2 + 2\rho\phi(t)V_i(t)\left(1-\mathrm{p(t)}\right)\right\}\pi(\mathrm{t})\mathrm{dt} \quad (16)$$

Therefore, the pension company's problem can now be written as

$$H(U;t{:}T) = Max_{\pi(t)}\, E\left[\cup\left(U^{\pi}(t)\right)\right]U(0) = U. \qquad (17)$$

Subject to

$$dU^{\pi}(t) = \left[[(\mu+k)-(\alpha+\lambda+r)]\pi(t) + ru(t) + \left(\theta-\eta p(t)\right)\left(\psi(t)V_i(t) - \zeta s_i(t) - C(t)\right)\right] + \pi(t)\beta dw_H^2(t) + (1-\mathrm{p(t)})\phi(t)V_i(t)dw_H^1(t),$$

In the case where transaction costs and taxes are charged only on risky investment and

$$H(U;t{:}T) = Max_{\pi(t)}\, E\left[\cup\left(U^{\pi}(t)\right)\right]U(0) = U. \qquad (19)$$

Subject to $dU^{\pi}(t) = [(\mu+k)-r]\pi(t) + [r-(\alpha+\lambda)]U(t) + \left(\theta-\eta p(t)\right)\phi(t)\psi(t)V_i(t) - \zeta s_i(t) - C(t)]dt + [\pi(t)\beta]^2 dw_H^2(t) + (1-\mathrm{p(t)})\phi(t)V_i(t)dw_H^1(t)$

In the case where transaction costs and taxes are charged on the pension fund company's total investment (both risky and risk free

OPTIMIZATION

We are going to look at the optimization program for the pension fund problem .Considering the optimization problem based on logarithmic utility function.

$$\frac{U \cup '(U)}{\cup '(U)} \qquad (20)$$

Where $\cup$ is the wealth level of the Company considered? We have the utility function of the type

$$\cup(U) = In(U) \qquad (21)$$

The exponential Utility function has absolute risk adverse parameter.

The aim of this work is to give explicit solutions to the pensioners' problems considering the assumptions given. The following mean-variance portfolio selection problem for pension fund problem becomes

$$G(U;t;T) = Max_{\pi(t)}\, E\left[\int_0^T e^{-\theta\tau}\left(InC(\tau)dt + e^{-\theta\tau}(InU_{\tau})\right)\right] \qquad (22)$$

Subject to

$$dU^{\pi}(t) = \left[[(\mu+k)-(\alpha+\lambda+r)]\pi(t) + ru(t) + \left(\theta-\eta p(t)\right)\left(\psi(t)V_i(t) - \zeta s_i(t) - C(t)\right)\right]dt + \pi(t)\beta dw_H^2(t) + (1-\mathrm{p(t)})\phi(t)V_i(t)dw_H^1(t)$$

In the case where transaction costs and taxes are charged on the pension fund total investment (both risky and risk free)

Theorem

The optimal strategy that maximize the logarithmic utility of pension fund at the terminal time T, when transaction cost and tax are charged on the risky investment only is to invest at time $t \leq (\tau)$

$$\pi_R^* = \frac{[(\mu + k) - (\alpha + \lambda + r)]U(t)}{\beta^2} + \frac{\rho(p(t) - 1)V_i(t)}{\beta}$$

With optimal reinsurance proportion

$$P_R^*(t) = 1 + \frac{\rho\beta\pi(t)}{V_i} - \frac{\eta(1 - \zeta s_i(t))}{{V_i}^2},$$

Optimal consumption

$$C_R^*(t) = U(t)e^{\int_t^T \varphi(\tau)d\tau},$$

and optimal value function;

$$G(U;t;T) = InU(t)e^{\int_t^T \varphi(\tau)d\tau},$$

Obtaining the Hamilton Jacobi –Bellman (HJB) partial differential equation we start with the Bellman equation thus:

$$G(U;t;T) = Max_{\pi(t)}E\left\{InC(t) + \frac{1}{1+\vartheta}E[G(U;t+\Delta t;T)]\right\} \quad (23)$$

Where U denoted the wealth of the pension fund at time $t + \Delta t$.

The actual utility and the time interval of length Δt is$InC(t)\Delta t$and the counting over time interval is $\frac{1}{1+\vartheta}\Delta t, where\ \vartheta > 0$.

Rewriting equation (23)

$$G(U;t;T) = Max_{\pi(t)}E\left\{InC(t)\Delta t + \frac{1}{1+\vartheta\Delta t}E[G(U;t+\Delta t;T)]\right\} \quad (24)$$

Multiplication of both sides of (24) by the factor $1 + \vartheta\Delta t$ and rearranging gives

$$\vartheta G\Delta t = Max_{\pi(t)}\{InC(t)\Delta t(1 + \vartheta\Delta t) + E[\Delta \mathrm{G}]\} \quad (25)$$

The division of 25 by the factor $1 + \vartheta\, \overline{\nabla}\, (t)$ and rearranging obtained

$$\vartheta G = Max_{\pi(t)}\left\{InC(t) + \frac{1}{dt}E[\mathrm{dG}]\right\} \quad (26)$$

Ito's lemma (Nile 25) which state that

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial U}dU + \frac{1}{2}\frac{\partial^2 G}{\partial U^2}(dU)^2 \ldots \quad (27)$$

Substituting in (27) ,the Ito Lemma for $dU^{\pi}(t)\ and\ \langle dU^{\pi}(t)\rangle$ using equation 12 and 16 obtained the stochastic differential equation (SDE)

$$
\begin{aligned}
InC(t) + G_t + G_U \Big[& [(\mu + k) - (\alpha + \lambda + r)]\pi(t) + ru(t) \\
& + \big(\theta - \eta p(t)\big)\big(\phi(t)V_i(t) - \zeta s_i(t) - C(t)\big)\Big] + \frac{1}{2}G_{UU}[\pi(t)\beta]^2 \\
& + (1 - p(t))^2\phi(t)V_i(t) + 2\rho\phi(t)V_i(t)\big(1 - p(t)\big)\pi(t) \\
& = 0 \qquad (28a)
\end{aligned}
$$

where

$$E\big(dw^1(t)\big) = E(dw^2(t)) = 0 \qquad (28b)$$

the application of the first order condition on (28a) with respect to consumption yield the optimal consumption as

$$C^*(t) = G_U^{-1}(U;t;T)(29)$$

differentiating (28a) with respect to $\pi(t)$ obtain the equation

$$G_U[(\mu + k) - (\alpha + \lambda + r)] + G_{UU}\pi(t)\beta^2 + \rho\phi(t)V_i(t)\big(1 - p(t)\big) = 0 \qquad (30)$$

this simplifies to obtaining the optimal investment in the risky asset as

$$\pi_R^* = \frac{\rho(p(t) - 1)}{\beta} - \frac{[(\mu + k) - (\lambda + \alpha + r)]}{\beta^2 G_{UU}} \qquad (31)$$

also differentiating 28a with respect to $p(t)$ gives

$$\eta\big(\psi(t)V_i(t) - \zeta s_i(t)\big) - [\phi(t)V_i(t)]^2(1 - p(t))G_{UU} - \rho\beta V_i(t)\pi G_{UU} = 0 \qquad (32)$$

and simplifies to

$$[\phi(t)V_i(t)]^2 G_{UU}p(t) = \eta(\phi(t)V_i(t) - \zeta \mathrm{s_i}(\mathrm{t}))G_u + [\phi(t)V_i(t)]^2 G_{UU} + \rho\beta V_i(t)\pi(t)G_{UU}$$

and from which the optimum reinsured proportion of the company wealth equals

$$p_R^*(t) = 1 + \frac{\rho\beta\pi(t)}{V_i} + \frac{\eta(\psi(t)V_i - \zeta s_i(t))}{V_i G_{UU}}G_U \qquad (33)$$

now let

$$G(U;t:T) = InU(t)F(t;T) \qquad (34)$$

such that at the terminal time T ,

$$F(T,T) = 1 \qquad (35)$$

be a classical solution of the pension fund problem then

$$\left.\begin{aligned} G_t = InU{:}\,F(t,T);\ G_U = \frac{1}{U}F(t{:}T) \\ G_{UU} = -\frac{1}{U^2}F(t{:}T) \end{aligned}\right\} \qquad (36)$$

Therefore using (36) in (29) the optimal consumption becomes

$$C_R^*(t) = \frac{U(t)}{F(t:T)},$$

$$= U(t)e^{\int_t^T \varphi(\tau)d\tau} \qquad (37)$$

And optimal investment in the risky asset

$$\pi_R^*(t) = \frac{\rho V_i(p(t)-1)}{\beta} - \frac{[(\alpha+\lambda+r)-(U+K)]}{\beta^2}U(t) \qquad (38)$$

Also ,the optimal reinsurance proportion of company's wealth is

$$p_R^*(t) = 1 + \frac{\rho\beta\pi}{V_i} - \frac{\eta\big(1-\Delta s_i(t)\big)}{V_i^{\,2}}U(t) \qquad (39)$$

Clearly equation 37 to equation 39 shows the optimal consumption, optimal investment in the risky asset and optimal reinsured proportion of the company's wealth are all horizon and wealth dependent.

Putting equation (36) to (28a) we obtain the ordinary differential equation

$$\begin{aligned} &InC(t) + InU.F'(t,T) \\ &\qquad + \frac{1}{U}F(t;T)\Big[[(\mu+k)-(\alpha+\lambda+r)]\pi(t) + rU(t) \\ &\qquad + \big(\theta-\eta p(t)\big)\big(\psi(t)V_i(t) - \zeta s_i(t) - C(t)\big)\Big] - \frac{1}{2U^2}F(t;T)[\pi(t)\beta]^2 \\ &\qquad + \big(1-p(t)\big)^2\phi(t)V_i(t) + 2\rho\beta V_i(t)\big(1-p(t)\big)\pi(t) = 0 \qquad (40) \end{aligned}$$

Let

$$InC(t) = \frac{m}{U(t)}F(t;T) \qquad (41)$$

equation (40) becomes

$$\begin{aligned} &InU.F'(t,T) + F(t;T)\Big[\frac{1}{U}[(\mu+k)-(\alpha+\lambda+r)]\pi(t) + rU(t) \\ &\qquad + \big(\theta-\eta p(t)\big)\big(\phi(t)V_i(t) - \zeta s_i(t) - C(t)\big)\Big] - \frac{1}{2U^2}F(t;T)[\pi(t)\beta]^2 \\ &\qquad + \big(1-p(t)\big)^2\phi(t)V_i(t) + 2\rho\beta V_i(t)\big(1-p(t)\big)\pi(t) = 0 \qquad (42) \end{aligned}$$

and reduce to

$$F'(t,T) + \varphi(t)F(t;T) = 0 \qquad (43a)$$

where

$$\begin{aligned} \varphi(t) = \frac{1}{U(t)}&\Big[\frac{1}{U}[(\mu+k)-(\alpha+\lambda+r)]\pi(t) + rU(t) + \big(\theta-\eta p(t)\big)\big(\psi(t)V_i(t) - \zeta s_i(t) - C(t)\big)\Big] \\ &- \frac{1}{2U^2}F(t;T)[\pi(t)\beta]^2 + \big(1-p(t)\big)^2\phi(t)V_i(t) \\ &+ 2\rho\beta V_i(t)\big(1-p(t)\big)\pi(t) \qquad (43b) \end{aligned}$$

From equation (43a) we obtain

$$\int_t^T \frac{F'(\tau:T)}{F(\tau:T)} d\tau = -\int_t^T \varphi(\tau) d\tau \qquad (44)$$

Which its integration gives

$$F(T;T) = e^{-\int_t^T \varphi(\tau) d\tau} \qquad (45)$$

And satisfies the boundary condition

$$F(T:T) = 1$$

Therefore ,the optimal value function of the pensionfund problem when transaction cost and tax are charged on the risky investment only is now given as

$$G(U;t;T) = InU(t)e^{-\int_t^T \varphi(\tau) d\tau} \qquad (46)$$

CASE 2: When transaction cost and taxes are charged on the pension fund total investment.

Theorem 2

The optimal policy that maximizes the expected logarithmic utility of a pension fund wealth at the terminal time T when transaction costs and taxes are charged on the company's total investment is to invest in the risky asset at each time t.

$$\pi_T^*(t) = \rho V_i \left[\frac{(p(t)-1)}{\beta}\right] - \left[\frac{r-(U+K)}{\beta^2}\right] U(t)$$

With optimal reinsured wealth

$$p_T^*(t) = 1 + \frac{\rho\beta\pi(t)}{V_i} - \frac{\eta\left(1-\Delta s_i(t)\right)}{{V_i}^2} U(t)$$

Optimal consumption

$$C_T^*(t) = U(t)e^{\int_t^T \varphi(\tau) d\tau}$$

And optimal value function

$$G_T^*(U;t;T) = InU(t)e^{-\int_t^T \varphi(\tau) d\tau}$$

Proof

Adopting steps of 14 and 16, obtain the HJB equation

$$\begin{aligned} InC(t) + G_t + G_U[(\mu+k)-r]\pi(t) + [r-(\alpha+\lambda)]U(t) \\ + \left(\theta - \eta p(t)\right)\left(\phi(t)V_i(t) - \zeta s_i(t) - C(t)\right) + G_{UU}[\pi(t)\beta]^2 + \left(1-p(t)\right)^2\phi(t)V_i(t) \\ + 2\rho\beta V_i(t)\left(1-p(t)\right)\pi(t) = 0 \qquad (47) \end{aligned}$$

Differentiating (47) with respect to $C(t)$,we obtain the optimal consumption

$$C_T^*(t) = G_U^{-1}(U; t\!:\!T) \tag{48}$$

The first order condition applied to 51 with respect to $\pi(t)$ as

$$\pi_T^* = \frac{\rho V_i[p(t)-1]}{\beta} - \frac{[r-(U+K)]G_U}{\beta^2 G_{UU}} \tag{49}$$

Differentiating (47) with respect to $p(t)$ obtains

$$-\eta\big(\phi(t)V_i(t)\big)G_U - [V_i(t)]^2 G_{UU} + [V_i(t)]^2 p(t) - \rho\beta V_i(t)\pi(t)G_{UU} = 0 \tag{50}$$

$$p_T^*(t) = 1 + \frac{\rho\beta\pi(t)}{V_i} - \frac{\eta\big(1-\Delta s_i(t)\big)G_U}{{V_i}^2 G_{UU}} \tag{51}$$

Applying 36 to 48,49 and 51 gives

The optimal investment in the risky asset as

$$\pi_T^*(t) = \frac{\rho V_i[p(t)-1]}{\beta} - \frac{[r-(U+K)]}{\beta^2}U(t) \tag{52}$$

Optimal consumption

$$C_T^* = \frac{U(t)}{F(t;T)} \tag{53}$$

and optimal reinsured proportion of the pension fund

$$p_T^*(t) = 1 + \frac{\rho\beta\pi(t)}{V_i} - \frac{\eta\big(1-\zeta s_i(t)\big)}{{V_i}^2}U(t) \tag{54}$$

which are dependent on horizon and wealth company 38 and 52 we obtain

$$\pi_R^*(t) = \frac{\rho V_i[p(t)-1]}{\beta} - \frac{[(\alpha+\lambda)+r)-(U+K)]}{\beta^2}U(t)$$

$$= \left[\frac{\rho V_i[1-p(t)]}{\beta} - \frac{[r-(U+K)]}{\beta^2}\right]U(t) - \frac{[\alpha+\lambda]}{\gamma\beta^2}U(t)$$

$$\pi_R^*(t) = \pi_T^*(t) - \frac{[\alpha+\lambda]}{\beta^2}U(t) \tag{55}$$

Equation (55) shows that charging transaction costs and taxes on the pensioners' fund will warrant an increment in the risky investment by fraction $\frac{[\alpha+\lambda]}{\beta^2}U(t)$ of his fund.

From 37,39,53,and 54.It is clear that charging transaction costs and taxes on the pension fund total fund or limiting them to the investment in the risky assets does not affect the reinsured proportion of the insurer's investments.

Now substituting (36) into (47)

$$\begin{aligned} &InC(t) + InU.F'(t,T) \\ &\qquad + F(t;T)\Big[[(\mu+k)-r]\pi(t) + [r-(\alpha+\lambda)]U \\ &\qquad + \big(\theta-\eta p(t)\big)\big(\phi(t)V_i(t) - \zeta s_i(t) - C(t)\big)\Big] - \frac{1}{U^2}F(t;T)[\pi(t)\beta]^2 \\ &\qquad + \big(1-p(t)\big)^2\phi(t)V_i(t) + 2\rho\beta V_i(t)\big(1-p(t)\big)\pi(t) = 0 \end{aligned}$$

which reduce to

$$F(t;T) + z(t)F(t;T) = H(t)(57a)$$

Where

$$\begin{aligned} z(t) = \frac{1}{UInU}\Big[[(\mu+k)-r]\pi(t) + [r-(\alpha+\lambda)]U + \big(\theta-\eta p(t)\big)\big(\phi(t)V_i(t) - \zeta s_i(t) - C(t)\big)\Big] \\ - \frac{1}{U^2InU}[\pi(t)\phi\beta]^2 + \big(1-p(t)\big)^2\phi(t)V_i(t) + 2\rho\beta V_i(t)\big(1-p(t)\big)\pi(t) \qquad (57b) \end{aligned}$$

And

$$H(t) = \frac{-InC(t)}{InU(t)}$$

The solution to (57a) is obtained using the theorem below

Theorem 3

If $z(t) and\ H(t)$ are continous function in the interval $1 = (t:T)$, then the general solution of $\frac{dF(t,T)}{dt} + z(t)F(t,T) = H$ in the interval I $= (t;T)$ is given by

$$F(t;T) = e^{-\int_t^T z(\tau)d\tau}\left[\int_t^T e^{\int_t^T z(s)ds}ds + \varphi\right] \quad (58)$$

(Myint,[26])

Therefore applying the boundary condition (35)

$$F(T;T) = \varphi = 1 \qquad (59)$$

And

$$F(t;T) = e^{-\int_t^T z(\tau)d\tau}\left[\int_t^T e^{\int_t^T z(s)ds}ds + 1\right](60)$$

the optimal value function of the pension fund problem is given as

$$G_T^*(U,t;T) = InU(t)e^{-\int_t^T \varphi(\tau)d\tau} \qquad (60)$$

which at terminal time T is

$G_T^*(U,t;T) = InU(T)$ as expected.

Sensitivity Analysis and Discussion

In this section, numerical simulations showing the relationship between the optimal strategy that maximize the logarithmic utility of pension fund and some sensitive parameters are presented. To achieve this, the following data are used similar to [21,30] unless otherwise stated: $\vartheta = 0.5$, $\theta = 0.01$, $\mathscr{B} = 0.01$, $\mathscr{R} = 0.1$, $\gamma = 0.5$, $x = 1$, $\rho = 0.1$, $\sigma = 0.1$, $\aleph = 0.01$,$\beta = 0.01$, $a = 0.01$, $n = 0.001$, $m_0 = 20$, $T = 30$.

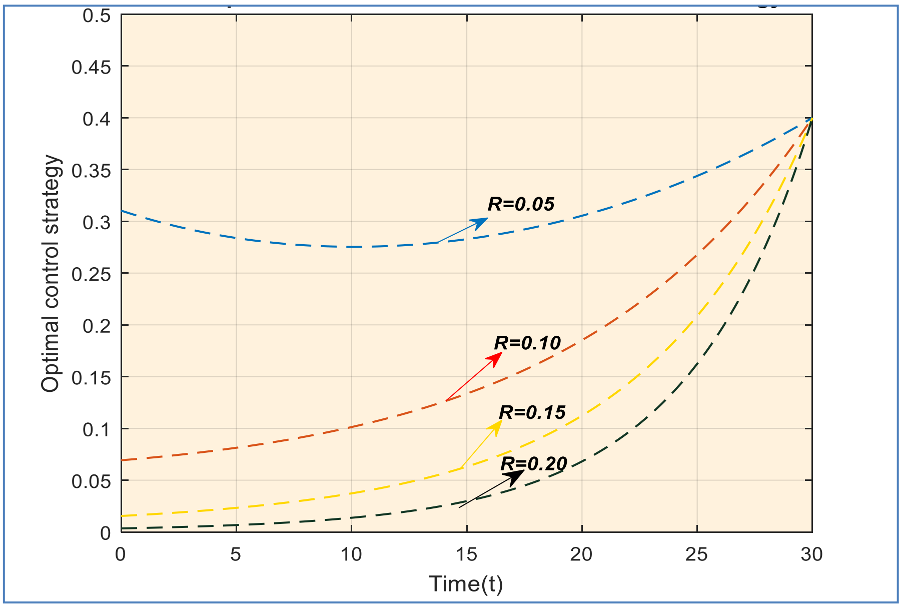

Fig 1: The impact of risk free interest rates on optimal strategy that maximize the logarithmic utility of pension fund

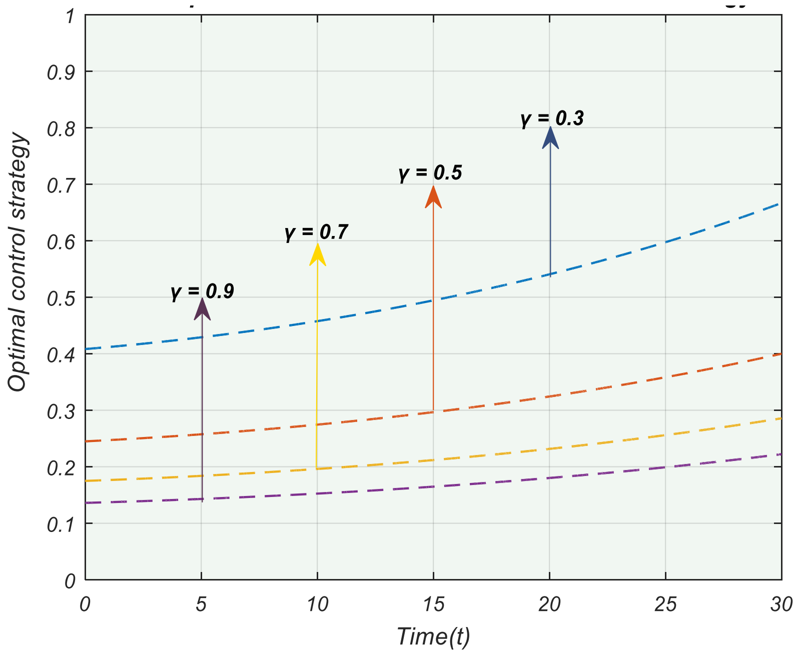

Fig 2: The impact of risk averse coefficient on optimal strategy that maximize the logarithmic utility of pension fund

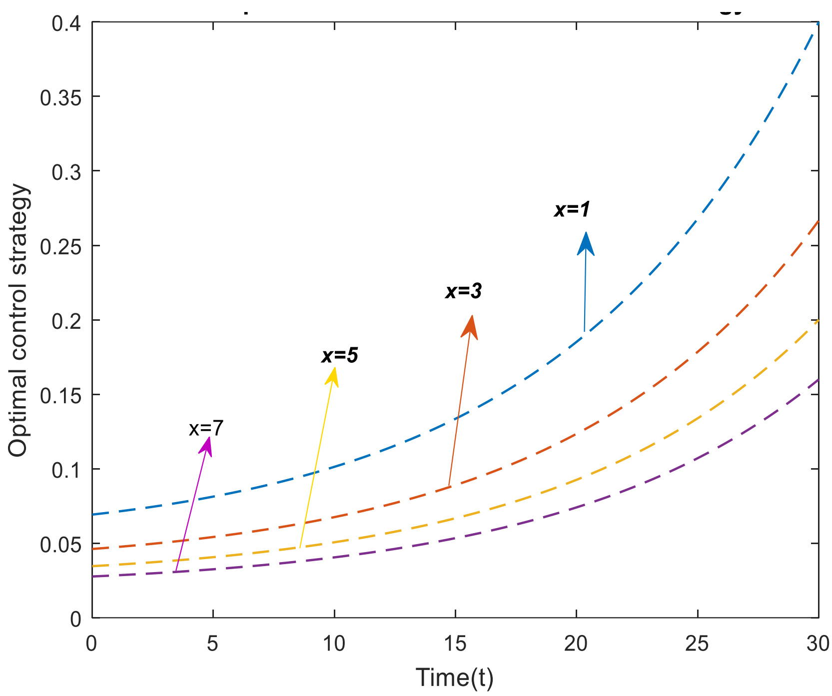

Fig 3: The impact of initial fund size onoptimal strategy that maximize the logarithmic utility of pension fund

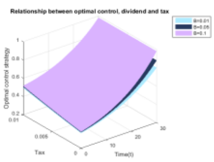

Fig 4: The relationship between optimal strategy that maximize the logarithmic utility of pension fund, dividend and tax

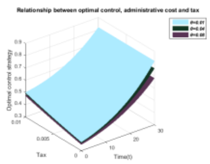

Fig 5: The relationship between optimal strategy that maximize the logarithmic utility of pension fund, transaction cost and tax

The efficient frontier which gives a relationship between the expectation and the variance shows that the pension fund return is directly proportional to the variance; the implication of this is that, pension fund administrators who are willing to invest in highly risky assets have higher chances of getting more returns at the end of the investment period. i.e more risk, implies higher expectation and vice versa. Also, from fig 1, optimal strategy that maximize the logarithmic utility of pension decrease as the risk free interest rate increases and increase when the risk free interest rate decreases. This simply indicates that pension fund administrators will likely want to invest in risky asset when the interest rate from the risk free asset is not attractive. However, if the risk free interest rate is attractive enough, This implies fund administrators will invest more in the risk free asset, thereby reducing their investment in the risky asset.

Figure 2, present a relationship between the optimal distribution strategy that maximize the logarithmic utility of pension fund of the risky asset and the correlation coefficient. It is observed that is inversely proportional to the correlation coefficient. Since the value is determined by the information generated by the two Brownian motions, the implication is that a higher value of the correlation coefficient shows how volatile and risky the stock market is at that time. Hence, this may lead to a reduction in the fraction of the pension fund wealth invested in the stock market.

Figure 3, discuss the effect of the risk aversion coefficient on the optimal strategy that maximize the logarithmic utility of pension fund and we observed that the optimal control strategy for the risky asset, is inversely proportional to the risk aversion coefficient parameter. What we deduced from the graph in figure 3 is that pension administrators may invest members fund with lesser percentage of their wealth in the risky asset (stock) while members with lower risk aversion coefficient may invest higher percentage of their wealth in the risky assets while reducing investment in the risk free asset.

Figure 4, discuss the effect of the initial fund size on the optimal strategy that maximize the logarithmic utility of pension fund and we observed that the optimal strategy that maximize the logarithmic utility of pension fund for the risky asset, is inversely proportional to the initial fund size parameter of the pension fund. The implication of figure 4 is that if the initial fund size at the time of investment is high, members may reduce the amount of risk to be taken, by reducing the proportion of his or her wealth to be invested in the risky asset and vice versa.

Figure 5, represents the relationship between the optimal strategy that maximize the logarithmic utility of pension fund, dividend and tax. We observed that the optimal strategy that maximizes the logarithmic utility of pension fund of the pension is an increasing function of the dividend and tax. The consequence of the graph in figure 5 is that if the dividend from the risky asset is attractive, the members may be tempted to invest more in risky asset and will invest less if the dividend from such investment is not attractive. Also, we observed that as the proportion of tax increase, the pension fund company tends to invest more in risky asset in other to balance the wealth of members before retirement. This shows that higher tax rate could expose a pension fund to higher risk during investment planning and decision making.

Figure 5, shows the relationship between the optimal strategy that maximize the logarithmic utility of pension fund, administrative (transaction cost) and tax. We observed that the optimal control strategy of the pension fund is a decreasing function of the administrative charges and an increasing function of the tax. The consequence of the graph in figure 5 is that if the administrative charges on investment of the risky asset is relatively low, the members may be encouraged to invest more in risky asset and may invest less if otherwise.

CONCLUSION

The model used has the basic claim process assumed to follow a fractional Brownian motion with drift. The pension fund traded in two assets; a risky asset and a risk free asset. The trading was done under dividends yields, transaction costs, tax, and consumption where, transaction costs and tax were considered in two perspectives; when they were charged on the risky asset only and when they were charged on the total investment of the company. Pension Fund Problem was established .An explicit solution to the pensioners' problems was derived under stated condition. It was found that charging, transaction costs and tax on the pension fund total investment increased the pension fund in the risky asset as compared to when, transaction cost and tax were charged on the risky asset only, by a fraction of the pension fund total wealth. These conditions did not alter the optimal reinsured proportion of the company's investment and consumption. The optimal strategies were found to be both horizon and wealth dependent and the condition for only horizon dependency obtained. Finally the optimal value functions for the pension fund expected logarithmic utility maximization for both cases considered were obtained.

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