The Fischer-Cliford Matrices and Character Table of The Group $2^6 : GL(4,2)$

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***Abstract*- The purpose of this paper is constructing the Fischer-Clifford matrices and the character tables for the group $2^6 : GL(4,2)$.**

***Keywords*: linear groups, group extensions, character table, Clifford theory, inertia groups, Fischer-Clifford matrix.**

I. Introduction

The theory of Clifford-Fischer matrices, which is based on Clifford Theory [1] which was developed by B. Fischer [5]. Let $\bar{G} = 2^6 : GL(4,2)$ be the split extension of $N = 2^6$ by $GL(4,2)$ where N is the vector space of dimension 6 over GF(2) on which G acts naturally. The aim of this paper is to construct the character table of $\bar{G}$ by using the technique of Fischer-Clifford matrix $M(g)$ for each class representative g of G and the character tables of the inertia factor groups H_i of the inertia groups $\overline{H_i} = 2^6 : H_i$. we use the properties of the Fischer-Clifford matrices discussed in ([2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [17], [18], [20]) to compute entries of these matrices.

The Fischer-Clifford matrix $M(g)$ will be portioned row-wise into blocks, where each block corresponds to an inertia group $\overline{H_i}$. Now using the columns of character table of the inertia factor H_i of $\overline{H_i}$ which correspond to classes of H_i which fuse to the class [g] in G and multiply these columns by the rows of the Fischer-Clifford matrix $M(g)$ that correspond to $\overline{H_i}$. Thus, we construct the portion of the character table of $\bar{G}$ which is in the block corresponding to $\overline{H_i}$ for the classes of $\bar{G}$ that come from the coset Ng. For more information about this technique see ([2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [17], [18], [20]). The character table of $\bar{G}$ will be divided row-wise into blocks where each block corresponding to an inertia group $\overline{H_i} = N : H_i$. The computations have been carried out with the aid of Maxima [14], MAGMA [13] and GAP [19] , Finally we will follow the notion of Atlas [16].

II. Theory of Fischer-Clifford Matrices

Let $\bar{G} = N : G$ be a split extension of N by G. Then for $\theta \in Irr(N)$, we define $\bar{H} = \{x \in \bar{G} : \theta^x = \theta\} = I_{\bar{G}}(\theta)$ and $H = \{x \in G : \theta^g = \theta\} = I_G(\theta)$ where $I_{\bar{G}}(\theta)$ is the stabilizer of θ in the action of $\bar{G}$ on Irr(N), we have that $I_{\bar{G}}(\theta)$ is a subgroup of $\bar{G}$ and N is normal subgroup in $I_{\bar{G}}(\theta)$. Also $[\bar{G} : I_{\bar{G}}(\theta)]$ is the size of the orbit containing θ . Then it can be shown that $\bar{H} = N : H$, where $\bar{H}$ is the inertia group of θ in $\bar{G}$. The inertia factor $\bar{H}/N \cong H$ can be regarded as the inertia group of θ in the factor group $\bar{G}/N \cong G$. Define θ^g by $\theta^g(n) = \theta(gng^{-1})$ for $g \in \bar{G}, n \in N$, $\theta^g \in Irr(N)$. We say that θ is extendible to $\bar{H}$ if there exists $\varphi \in Irr(\bar{H})$ such that $\downarrow_N = \theta$.

If θ is extendible to $\bar{H}$, then by Gallagher [15], we have $\{\alpha : \alpha \in Irr(\bar{H}), < \alpha \downarrow_N, \theta > \neq 0\} = \{\beta\varphi : \beta \in Irr(\bar{H}/N)\}$. Let $\bar{G}$ has the property that every irreducible character of N can be extended to its inertia group. Now let $\theta_1 = 1_N, \theta_2, \dots, \theta_t$ be representatives of the orbits of $\bar{G}$ on Irr(N), $\overline{H_i} = I_{\bar{G}}(\varphi_i), 1 \le i \le t, \varphi_i \in Irr(\overline{H_i})$ be an extension of θ_i to $\overline{H_i}$ and $\beta \in Irr(\overline{H_i})$ such that $N \subseteq Ker(\beta)$. Then it can be shown that

$$Irr(\bar{G}) = \bigcup_{i=1}^{t}\{(\beta\varphi_i)^{\bar{G}} : \beta \in Irr(\overline{H_i}), N \subseteq Ker(\beta)\} = \bigcup_{i=1}^{t}\{(\beta\varphi_i)^{\bar{G}} : \beta \in (\overline{H_i}/N)\}$$

Hence the irreducible characters of $\bar{G}$ will be divided into blocks, where each block corresponds to an inertia group $\overline{H_i}$.

Let $\overline{H_i}$ be the inertia factor group and φ_i be an extension of θ_i to $\overline{H_i}$. Take $\theta_1 = 1_N$ as the identity character of N, then $\overline{H_1} = \bar{G}$ and $H_1 \cong G$. Let $X(g) = \{x_1, x_2, \dots, x_{c(g)}\}$ be a set of representatives of the conjugacy classes of $\bar{G}$ from the coset $N\bar{g}$ whose images under the natural homomorphism $\bar{G} \to G$ are in [g] and we take $x_1 = \bar{g}$. We define

$$R(g) = \{(i, y_k) : 1 \le i \le t,\ H_i \cap [g] \ne 0, 1 \le k \le r\}$$

and we note that y_k runs over representatives of the conjugacy classes of elements of H_i which fuse into [g] in G. Then we define the Fischer-Clifford matrix M(g) by $M(g) = \left(a^j_{(i,y_k)}\right)$, where $a^j_{(i,y_k)} = \sum_l^t \frac{|C_{\bar{G}}(x_j)|}{|C(y_{lk})|}\varphi_i(y_{lk})$ with columns indexed by X(g) and rows indexed by R(g) and where $\sum_l^t$ is the summation over all l for which $y_{lk} \sim x_j$ in $\bar{G}$. Then the partial character table of $\bar{G}$ on the classes $\{x_1, x_2, \dots, x_{c(g)}\}$ is given by $\begin{bmatrix} C_1(g)M_1(g) \\ C_2(g)M_2(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$ where the Fischer-Clifford matrix $M(g) = \begin{bmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{bmatrix}$ is divided into blocks $M(g)$ with each block corresponding to an inertia group $\overline{H_i}$ and $C_i(g)$ is the partial character table of H_i consisting of the columns corresponding to the classes that fuse into [g] in G.

We can also observe that the number of irreducible characters of $\bar{G}$ is the sum of the number of irreducible characters of the inertia factors $H_i's$. The group $\bar{G} = 2^6{:}GL(4,2)$ is a split extension with 2^6 abelian and therefore by Mackey's theorem (see [8] - Theorem 4.1.12) , we have each irreducible character of 2^6 can be extended to its inertia group in $\bar{G}$. Hence by the above theoretical outline we can fully determine the character table of $\bar{G} = 2^6{:}GL(4,2)$.

III. The conjugacy classes of $2^6{:}GL(4,2)$

In this section, we will use the method of coset analysis to determine the conjugacy classes of $\bar{G} = 2^6{:}GL(4,2)$. We refer the reader to ([2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [17], [18], [20]) for full details and background material regarding the method of coset analysis. Most of the information, which involved the conjugacy classes and permutation characters, were obtained by using direct computations in GAP [19] and MAGMA [13].

The general linear group $GL(4,2)$ of order =20160 is a subgroup of the general linear group $GL(5,2)$. By MAGMA [13], We can generate the group GL(4,2) by the two 6×6 matrices:

$$a = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, \quad b = \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&1&0&0\\ 0&0&1&0&0&0 \end{bmatrix}$$

We construct the conjugacy class representative of G in terms of 6×6 matrices over GF(2) by GAP[19].

The GL(4,2) has 14 conjugacy classes and under the action of GL(4,2) on 2^6 , we obtain three orbits of lengths 1 , 35 and 28. The point stabilizers for orbits of lengths 1, 35 and 28 are the subgroups GL(4,2) , $(A_4 \times A_4) : 2^2$ and S_6 of indices 1, 35 and 28, respectively in GL(4,2). Let $\mathcal{X}(GL(4,2)|2^6)$ be the permutation character of GL(4,2) on 2^6. The values of $\mathcal{X}(GL(4,2)|2^6)$ on different classes of GL(4, 2) determine the number of k of fixed points of each conjugacy class of GL(4,2) in 2^6. These values of the $k's$ will help us to calculate the conjugacy classes of $6{:}GL(4,2)$ which are listed in Table 1. Consequentially, having obtained the values of the $k's$ for the various classes of G, we then need to calculate the values of $f_i's$ corresponding to the various $'s$, where $f_i's$ are the number of orbits $Q_i's$ for $1 \leq i \leq k$, that fuse together under the action of $C_G(g)$ to form one orbit Δ_j. For this purpose, we used Programme A [8]. For a class representative $dg \in \bar{G}$, where $d \in 2^5$, $g \in GL(4,2)$ and $o(g) = m$, by Theorem 3.3.10 in [20] we have

$$o(dg) = \begin{cases} m & if \quad w = 1_N \\ 2m & otherwise \end{cases}$$

To calculate the orders of the class representative $dg \in \bar{G}$, we used Programme B [8]. If $o(g) = m$ and $w = 1_N$ then $o(dg) = 2m$ otherwise if $\neq 1_N$, then $o(dg) = 2m$. To each class of $\bar{G}$, we have attached some weight m_{ij} which will be used later in computing the Fischer matrices of the extension. These weights are computed by the formula

$$m_{ij} = [N_{\bar{G}}(N\bar{g}_i){:}C_{\bar{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\bar{G}}(g_{ij})|}$$

Thus, we obtained 41 conjugacy classes for the group $2^6{:}GL(4,2)$ and we list these result about the conjugacy classes of $\bar{G}$ in Table 1.

Table (1): Conjugacy Classes of $2^6{:}GL(4,2)$

$g_i \in G$	k_i	f_i	$m_{i,j}$	d_i	w	$[x]_{\bar{G}}$	$\|[x]_{\bar{G}}\|$	$\|C_{\bar{G}}(x)\|$
1A	64	1	1	(0,0,0,0,0,0)	(0,0,0,0,0,0)	1a	1	1290240
		35	35	(0,0,0,0,0,1)	(0,0,0,0,0,1)	2a	35	36864
		28	28	(0,0,1,1,0,0)	(0,0,1,1,0,0)	2b	28	46080
7A	1	1	64	(0,0,0,0,0,0)	(0,0,0,0,0,0)	7a	184320	7
7B	1	1	64	(0,0,0,0,0,0)	(0,0,0,0,0,0)	7b	184320	7
3A	4	1	16	(0,0,0,0,0,0)	(0,0,0,0,0,0)	3a	17920	72
		1	16	(0,0,0,0,0,1)	(0,0,1,0,1,0)	6a	17920	72
		1	16	(0,0,0,1,0,0)	(1,0,0,1,0,1)	6b	17920	72
		1	16	(0,0,0,1,0,1)	(1,0,1,1,1,1)	6c	17920	72
3B	16	1	4	(0,0,0,0,0,0)	(0,0,0,0,0,0)	3b	448	2880
		5	20	(0,0,0,1,1,0)	(0,0,0,1,1,0)	6d	2240	576
		10	40	(0,0,0,0,0,1)	(0,1,1,1,0,1)	6e	4480	288
2A	16	1	4	(0,0,0,0,0,0)	(0,0,0,0,0,0)	2c	840	1536
		1	4	(0,0,1,1,0,0)	(1,0,1,1,1,1)	4a	840	1536
		3	12	(0,0,0,0,1,0)	(1,0,1,1,1,1)	4b	2520	512
		3	12	(0,0,0,1,1,1)	(0,0,0,0,0,0)	2d	2520	512
		8	32	(0,0,0,0,0,1)	(1,1,0,1,0,1)	4c	6720	192
2B	16	1	4	(0,0,0,0,0,0)	(0,0,0,0,0,0)	2e	420	3072
		1	4	(0,0,0,1,1,1)	(0,0,0,0,0,0)	2f	420	3072
		1	4	(0,0,1,0,0,1)	(0,0,0,0,0,0)	2g	420	3072
		1	4	(0,0,1,1,1,0)	(0,0,0,0,0,0)	2h	420	3072
		12	48	(0,0,0,0,0,1)	(0,1,1,0,0,0)	4d	5040	256
4A	4	1	16	(0,0,0,0,0,0)	(0,0,0,0,0,0)	4e	20160	64
		1	16	(0,0,0,0,0,1)	(0,0,0,0,0,0)	4f	20160	64
		1	16	(0,0,0,0,1,0)	(0,0,0,0,0,0)	4g	20160	64
		1	16	(0,0,0,0,1,1)	(0,0,0,0,0,0)	4h	20160	64
6A	4	1	16	(0,0,0,0,0,0)	(0,0,0,0,0,0)	6f	53760	24

		1	16	(0,0,0,0,0,1)	(0,0,0,0,0,0)	6g	53760	24
		1	16	(0,0,0,1,0,0)	(0,0,0,0,0,0)	6h	53760	24
		1	16	(0,0,0,1,0,1)	(0,0,0,0,0,0)	6i	53760	24
5A	4	1	16	(0,0,0,0,0,0)	(0,0,0,0,0,0)	5a	43008	30
		3	48	(0,0,0,0,0,1)	(0,0,1,1,1,1)	10a	43008	30
15A	1	1	64	(0,0,0,0,0,0)	(0,0,0,0,0,0)	15a	86016	15
15B	1	1	64	(0,0,0,0,0,0)	(0,0,0,0,0,0)	15b	86016	15
6B	4	1	16	(0,0,0,0,0,0)	(0,0,0,0,0,0)	6j	26880	48
		1	16	(0,0,0,1,0,0)	(1,0,0,1,1,0)	12a	26880	48
		2	32	(0,0,0,0,0,1)	(1,0,1,0,0,1)	12b	53760	24
4B	4	1	16	(0,0,0,0,0,0)	(0,0,0,0,0,0)	4i	40320	32
		1	16	(0,0,0,0,0,1)	(0,0,0,0,0,0)	4j	40320	32
		1	16	(0,0,0,1,0,0)	(0,0,1,0,0,0)	8a	40320	32
		1	16	(0,0,0,1,0,1)	(0,0,1,0,0,0)	8b	40320	32

Thus, the group $2^6{:}GL(4,2)$ has 41 conjugacy classes.

IV. The Inertia Factor Group of $\bar{G} = 2^6{:}GL(4,2)$

The action of G on N produce three orbits of lengths 1, 35 and 28. Hence by Brauer's theorem (see Theorem 5.1.4 in [20]) G acts on Irr(N) with the same number of orbits. The lengths of the these orbits will be $1, r, s, p$ where $1 + r + s + p = 64$, with corresponding point stabilizers H_1, H_2 and H_3 as subgroups of G such that $[G{:}H_1] = 1$, $[G{:}H_2] = 35$ and $[G{:}H_3] = 28$. Considering the indices of these subgroups in G and investigating the maximal and submaximal subgroups of G, we have the Group $GL(4,2)$ acting on Irr(2^6) produce three inertia factor groups:

(1) $H_1 = GL(4,2)$ of index equal 1,

(2) $H_2 = (A_4 \times A_4) : 2^2$ of index equal 35,

(3) $H_3 = S_6$ of index equal to 28.

Using GAP [19], we can generate the group H_2 in terms of 6×6 matrices over GF(2) by the follows matrices:

$$a_1 = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, a_2 = \begin{bmatrix} 1&1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{bmatrix}, a_3 = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},$$

$$a_4 = \begin{bmatrix} 1&1&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&1\\ 0&0&0&0&0&1 \end{bmatrix}, a_5 = \begin{bmatrix} 1&0&0&1&1&0\\ 0&0&1&0&1&1\\ 0&1&1&1&1&0\\ 0&0&1&0&0&1\\ 0&1&1&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix}$$

By using these generators in GAP [19], we obtained the conjugacy classes and the character tables for this group.

We can complete the fusion maps by using matrix conjugation into the group (4,2) . This fusion map is listed in table 2.

Table 2: The Fusion of the group $H_2 = (A_4 \times A_4) : 2^2$ into the group $G = GL(4,2)$

$[g]_{H_2}$	$\Rightarrow$	$[g]_G$	$[g]_{H_2}$	$\Rightarrow$	$[g]_G$	$[g]_{H_2}$	$\Rightarrow$	$[g]_G$
1a	$\Rightarrow$	1a	2a	$\Rightarrow$	2b	2b	$\Rightarrow$	2a
3a	$\Rightarrow$	3a	2c	$\Rightarrow$	2b	4a	$\Rightarrow$	4a

3b	⟹	3a	6a	⟹	6b	3c	⟹	3b
6b	⟹	6a	2d	⟹	2b	4b	⟹	4a
6c	⟹	6a	2e	⟹	2a	4c	⟹	4a
4d	⟹	4b						

Also, by using GAP [19], we can generate the group $H_3 = S_6$ in terms of 6×6 matrices over GF(2) by the follows matrices:

$$b_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$b_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, b_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

By using these generators in GAP [19], we obtained the conjugacy classes and the character tables for this group.

We can complete the fusion maps by using matrix conjugation in the group (4,2) . This fusion map is listed in table 3.

Table 3: The Fusion of the group $H_3 = S_6$ into the group $G = GL(4, 2)$

$[g]_{H_3}$	⟹	$[g]_G$	$[g]_{H_3}$	⟹	$[g]_G$	$[g]_{H_3}$	⟹	$[g]_G$
1a	⟹	1a	2a	⟹	2a	2b	⟹	2a
2c	⟹	2b	3a	⟹	3b	6a	⟹	6b
6b	⟹	3a	4a	⟹	4b	4b	⟹	4b
5a	⟹	5a	6b	⟹	6a			

V. The Fischer-Clifford Matrices of $\bar{G} = 2^6{:}GL(4,2)$

For each conjugacy class [g] of G with representative $g \in G$, we construct the corresponding Fischer-Clifford matrix $M(g)$ of $\bar{G} = 2^6{:}GL(4,2)$. We use the properties of the Fischer-Clifford matrices (see [2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [17], [18], [20]) together with fusions of H_2 into H_1 and H_3 into H_1 (Table 2, Table 3) to compute the entries of these matrices and to construct an algebraic system of linear and non-linear equations with the help of Maxima [14], we can solve these system of equations and compute all the Fischer matrices of $\bar{G}$.

The Fischer-Clifford matrix will be partitioned row-wise into blocks, where each block corresponding to an inertia group $\overline{H_\iota}$. We list the Fischer-Clifford matrices of $\bar{G}$ in Table 4:

Table 4: The Fischer-Clifford Matrices of $2^6{:}GL(4,2)$:

$M(g)$	$M(g)$
$M(1A) = \begin{bmatrix} 1 & 1 & 1 \\ 35 & 3 & -5 \\ 28 & -4 & 4 \end{bmatrix}$	$M(7A) = [1]$

$M(7B) = [1]$	$M(3A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$
$M(3B) = \begin{bmatrix} 1 & 1 & 1 \\ 5 & -3 & 1 \\ 10 & 2 & -2 \end{bmatrix}$	$M(2A) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 6 & -6 & 2 & -2 & 0 \\ 2 & -2 & -2 & 2 & 0 \\ 6 & 6 & -2 & -2 & 0 \end{bmatrix}$
$M(2B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & -1 \\ 4 & 4 & -4 & -4 & 0 \\ 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & -4 & 4 & 0 \end{bmatrix}$	$M(4A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$
$M(6A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$	$M(5A) = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$
$M(15A) = [1]$	$M(15B) = [1]$
$M(6B) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{bmatrix}$	$M(4B) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

VI. The Character Table of $\bar{G} = 2^6{:}GL(4,2)$

Now, we have:

(1) The conjugacy classes of $\bar{G} = 2^6{:}GL(4,2)$ (Table 1);

(2) The character tables of all the inertia factors (by using GAP [19]);

(3) The fusions of conjugacy classes of the inertia factors into classes of GL(4,2) (Table 2 & Table 3) ;

(4) The Fischer matrices of classes of $\bar{G} = 2^6{:}GL(4,2)$ (Table 4);

Thus, we can construct the character table of the $2^6{:}GL(4,2)$ as follows:

Table 5. The Character Table of $\bar{G} = 2^6{:}GL(4,2)$

Class	1a	2a	2b	7a	7b	3a	6a	6b	6c
Order	1	2	2	7	7	3	6	6	6
Size	1	35	28	184320	184320	17920	17920	17920	17920
X1	1	1	1	1	1	1	1	1	1
X2	7	7	7	0	0	1	1	1	1
X3	14	14	14	0	0	2	2	2	2
X4	20	20	20	-1	-1	-1	-1	-1	-1
X5	21	21	21	0	0	0	0	0	0
X6	21	21	21	0	0	0	0	0	0
X7	21	21	21	0	0	0	0	0	0
X8	28	28	28	0	0	1	1	1	1
X9	35	35	35	0	0	2	2	2	2
X10	45	45	45	A	/A	0	0	0	0
X11	45	45	45	/A	A	0	0	0	0
X12	56	56	56	0	0	-1	-1	-1	-1
X13	64	64	64	1	1	-2	-2	-2	-2
X14	70	70	70	0	0	1	1	1	1
X15	35	3	-5	0	0	2	0	-2	0
X16	35	3	-5	0	0	2	0	-2	0
X17	35	3	-5	0	0	2	0	-2	0
X18	35	3	-5	0	0	2	0	-2	0
X19	70	6	-10	0	0	1	3	-1	-3
X20	70	6	-10	0	0	1	3	-1	-3
X21	70	6	-10	0	0	1	-3	-1	3
X22	70	6	-10	0	0	1	-3	-1	3

X23	140	12	-20	0	0	-4	0	4	0
X24	210	18	-30	0	0	0	0	0	0
X25	210	18	-30	0	0	0	0	0	0
X26	315	27	-45	0	0	0	0	0	0
X27	315	27	-45	0	0	0	0	0	0
X28	315	27	-45	0	0	0	0	0	0
X29	315	27	-45	0	0	0	0	0	0
X30	420	36	-60	0	0	0	0	0	0
X31	28	-4	4	0	0	1	-1	1	-1
X32	28	-4	4	0	0	1	-1	1	-1
X33	140	-20	20	0	0	-1	1	-1	1
X34	140	-20	20	0	0	-1	1	-1	1
X35	140	-20	20	0	0	2	-2	2	-2
X36	140	-20	20	0	0	2	-2	2	-2
X37	252	-36	36	0	0	0	0	0	0
X38	252	-36	36	0	0	0	0	0	0
X39	280	-40	40	0	0	1	-1	1	-1
X40	280	-40	40	0	0	1	-1	1	-1
X41	448	-64	64	0	0	-2	2	-2	2

Class	3b	6d	6e	2c	4a	4b	2d	4c
Order	3	6	6	2	4	4	2	4
Size	448	2240	4480	840	840	2520	2520	6720
X1	1	1	1	1	1	1	1	1
X2	4	4	4	3	3	3	3	3
X3	-1	-1	-1	2	2	2	2	2
X4	5	5	5	4	4	4	4	4
X5	6	6	6	1	1	1	1	1
X6	-3	-3	-3	1	1	1	1	1
X7	-3	-3	-3	1	1	1	1	1
X8	1	1	1	4	4	4	4	4
X9	5	5	5	-5	-5	-5	-5	-5
X10	0	0	0	-3	-3	-3	-3	-3
X11	0	0	0	-3	-3	-3	-3	-3
X12	-4	-4	-4	0	0	0	0	0
X13	4	4	4	0	0	0	0	0
X14	-5	-5	-5	2	2	2	2	2
X15	5	-3	1	7	-5	3	-1	-1
X16	5	-3	1	7	-5	3	-1	-1
X17	5	-3	1	-5	7	-1	3	-1
X18	5	-3	1	-5	7	-1	3	-1
X19	-5	3	-1	2	2	2	2	-2
X20	-5	3	-1	2	2	2	2	-2
X21	-5	3	-1	2	2	2	2	-2
X22	-5	3	-1	2	2	2	2	-2
X23	5	-3	1	4	4	4	4	-4
X24	15	-9	3	-10	14	-2	6	-2
X25	15	-9	3	14	-10	6	-2	-2
X26	0	0	0	3	-9	-1	-5	3
X27	0	0	0	-9	3	-5	-1	3
X28	0	0	0	-9	3	-5	-1	3
X29	0	0	0	3	-9	-1	-5	3
X30	-15	9	-3	4	4	4	4	-4
X31	10	2	-2	8	4	-4	0	0
X32	10	2	-2	4	8	0	-4	0
X33	20	4	-4	0	12	4	-8	0
X34	20	4	-4	12	0	-8	4	0
X35	-10	-2	2	4	8	0	-4	0
X36	-10	-2	2	8	4	-4	0	0
X37	0	0	0	0	12	4	-8	0
X38	0	0	0	12	0	-8	4	0
X39	10	2	-2	-16	-8	8	0	0
X40	10	2	-2	-8	-16	0	8	0
X41	-20	-4	4	0	0	0	0	0

Class	2e	2f	2g	2h	4d	4e	4f	4g	4h
Order	2	2	2	2	4	4	4	4	4
Size	420	420	420	420	5040	5040	5040	5040	5040
X1	1	1	1	1	1	1	1	1	1
X2	-1	-1	-1	-1	-1	-1	-1	-1	-1
X3	6	6	6	6	6	2	2	2	2

X4	4	4	4	4	4	0	0	0	0
X5	-3	-3	-3	-3	-3	1	1	1	1
X6	-3	-3	-3	-3	-3	1	1	1	1
X7	-3	-3	-3	-3	-3	1	1	1	1
X8	-4	-4	-4	-4	-4	0	0	0	0
X9	3	3	3	3	3	-1	-1	-1	-1
X10	-3	-3	-3	-3	-3	1	1	1	1
X11	-3	-3	-3	-3	-3	1	1	1	1
X12	8	8	8	8	8	0	0	0	0
X13	0	0	0	0	0	0	0	0	0
X14	-2	-2	-2	-2	-2	-2	-2	-2	-2
X15	11	3	3	-5	-1	3	-1	-1	-1
X16	-5	3	3	11	-1	-1	-1	3	-1
X17	3	-5	11	3	-1	-1	-1	-1	3
X18	3	11	-5	3	-1	-1	3	-1	-1
X19	14	14	-2	-2	-2	2	2	-2	-2
X20	-2	-2	14	14	-2	-2	-2	2	2
X21	-2	14	-2	14	-2	-2	2	2	-2
X22	14	-2	14	-2	-2	2	-2	-2	2
X23	12	12	12	12	-4	0	0	0	0
X24	-6	-6	-6	-6	2	2	-2	2	-2
X25	-6	-6	-6	-6	2	-2	2	-2	2
X26	-21	3	3	27	-1	3	-1	-1	-1
X27	3	-21	27	3	-1	-1	3	-1	-1
X28	3	27	-21	3	-1	-1	-1	-1	3
X29	27	3	3	-21	-1	-1	-1	3	-1
X30	-12	-12	-12	-12	4	0	0	0	0
X31	4	-4	-4	4	0	0	0	0	0
X32	-4	4	4	-4	0	0	0	0	0
X33	4	-4	-4	4	0	0	0	0	0
X34	-4	4	4	-4	0	0	0	0	0
X35	12	-12	-12	12	0	0	0	0	0
X36	-12	12	12	-12	0	0	0	0	0
X37	-12	12	12	-12	0	0	0	0	0
X38	12	-12	-12	12	0	0	0	0	0
X39	8	-8	-8	8	0	0	0	0	0
X40	-8	8	8	-8	0	0	0	0	0
X41	0	0	0	0	0	0	0	0	0

Class	6f	6g	6h	6i	5a	10a	15a	15b
Order	6	6	6	6	5	10	15	15
Size	13440	13440	13440	13440	43008	43008	86016	86016
X1	1	1	1	1	1	1	1	1
X2	-1	-1	-1	-1	2	2	-1	-1
X3	0	0	0	0	-1	-1	-1	-1
X4	1	1	1	1	0	0	0	0
X5	0	0	0	0	1	1	1	1
X6	0	0	0	0	1	1	B	/B
X7	0	0	0	0	1	1	/B	B
X8	-1	-1	-1	-1	-2	-2	1	1
X9	0	0	0	0	0	0	0	0
X10	0	0	0	0	0	0	0	0
X11	0	0	0	0	0	0	0	0
X12	-1	-1	-1	-1	1	1	1	1
X13	0	0	0	0	-1	-1	-1	-1
X14	1	1	1	1	0	0	0	0
X15	2	0	-2	0	0	0	0	0
X16	-2	0	2	0	0	0	0	0
X17	0	-2	0	2	0	0	0	0
X18	0	2	0	-2	0	0	0	0
X19	-1	-1	1	1	0	0	0	0
X20	1	1	-1	-1	0	0	0	0
X21	1	-1	-1	1	0	0	0	0
X22	-1	1	1	-1	0	0	0	0
X23	0	0	0	0	0	0	0	0
X24	0	0	0	0	0	0	0	0
X25	0	0	0	0	0	0	0	0
X26	0	0	0	0	0	0	0	0
X27	0	0	0	0	0	0	0	0
X28	0	0	0	0	0	0	0	0
X29	0	0	0	0	0	0	0	0
X30	0	0	0	0	0	0	0	0
X31	1	-1	1	-1	3	-1	0	0

X32	-1	1	-1	1	3	-1	0	0
X33	1	-1	1	-1	0	0	0	0
X34	-1	1	-1	1	0	0	0	0
X35	0	0	0	0	0	0	0	0
X36	0	0	0	0	0	0	0	0
X37	0	0	0	0	-3	1	0	0
X38	0	0	0	0	-3	1	0	0
X39	-1	1	-1	1	0	0	0	0
X40	1	-1	1	-1	0	0	0	0
X41	0	0	0	0	3	-1	0	0

Class	6i	12a	12b	4i	4j	8a	8b
Order	6	12	12	4	4	8	8
Size	26880	26880	26880	40320	40320	40320	40320
X1	1	1	1	1	1	1	1
X2	0	0	0	1	1	1	1
X3	-1	-1	-1	0	0	0	0
X4	1	1	1	0	0	0	0
X5	-2	-2	-2	-1	-1	-1	-1
X6	1	1	1	-1	-1	-1	-1
X7	1	1	1	-1	-1	-1	-1
X8	1	1	1	0	0	0	0
X9	1	1	1	-1	-1	-1	-1
X10	0	0	0	1	1	1	1
X11	0	0	0	1	1	1	1
X12	0	0	0	0	0	0	0
X13	0	0	0	0	0	0	0
X14	-1	-1	-1	0	0	0	0
X15	1	1	-1	1	1	-1	-1
X16	1	1	-1	1	1	-1	-1
X17	1	1	-1	-1	-1	1	1
X18	1	1	-1	-1	-1	1	1
X19	-1	-1	1	0	0	0	0
X20	-1	-1	1	0	0	0	0
X21	-1	-1	1	0	0	0	0
X22	-1	-1	1	0	0	0	0
X23	1	1	-1	0	0	0	0
X24	-1	-1	1	0	0	0	0
X25	-1	-1	1	0	0	0	0
X26	0	0	0	-1	-1	1	1
X27	0	0	0	1	1	-1	-1
X28	0	0	0	1	1	-1	-1
X29	0	0	0	-1	-1	1	1
X30	1	1	-1	0	0	0	0
X31	2	-2	0	2	-2	0	0
X32	-2	2	0	0	0	2	-2
X33	0	0	0	-2	2	0	0
X34	0	0	0	0	0	-2	2
X35	-2	2	0	0	0	-2	2
X36	2	-2	0	-2	2	0	0
X37	0	0	0	2	-2	0	0
X38	0	0	0	0	0	2	-2
X39	2	-2	0	0	0	0	0
X40	2	-2	0	0	0	0	0
X41	0	0	0	0	0	0	0

Explanation of Character Value Symbols:

$A = \frac{-1-\sqrt{-7}}{2}$, $B = \frac{-1-\sqrt{-15}}{2}$

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