The imprimitive subgroups GL(3,3)

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Abstract

In this paper we determine the imprimitive subgroups of $GL(3,3)$.

1.Introduction:

Let M be the **JS**-imprimitive of $GL(3,3)$, that is, $M := GL(1,3)\ wr\ S_3$.

This group has order 48, and is generated by the matrices

$$c := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad a := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, b := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and

In order to obtain a polycyclic presentation for M we introduce the element

$$e := d^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad d := c^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad and$$

Then a polycyclic presentation for M is

$\{a, b, c, d, e | \quad a^2 = I_3 \quad ,$

$$b^a = b^2 \;, b^3 = I_3,$$
$$c^a = d \;, c^b = d \;, c^2 = I_3 \;,$$
$$d^a = c \;, d^b = e \;, d^c = d, d^2 = I_3,$$
$$e^a = e \;, e^b = e \;, e^c = e \;, e^d = e \;, e^2 = I_3\}.$$

Note that $M = < cde > \times < a,b,ce,de > \cong C_2 \times S_4$.

2. Theorems:

2.1-Theorem: A complete and irredundant list of $GL(3,5)$ – conjugacy class representatives of the irreducible subgroups of M is:

$< a,b,c,d,e >, \quad \cong C_2 \times S_4;$

$< a,b,ce,de >, \quad \cong S_4;$

$< acde,b,ce,de >, \cong S_4;$

$< b,c,d,e >, \quad \cong C_2 \times A_4$

$< b,ce,de >, \quad \cong A_4.$

proof: See Razzaghmaneshi (2002, 56)

Now let M be the **JS**-imprimitive of $GL(3,5)$, that is ,

$S_3.\ M := GL(1,5)\ wr$

This group has order 384, and is generated by the matrices

$$c := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, b := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and

In order to obtain a polycyclic presetation for M we introduce the elements

$$d := c^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $e := d^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Then a polycyclic presentation for M is

$\{a,b,c,d,e \mid a^2 = I_3 \;,$

$$b^a = b^2, b^3 = I_3,$$
$$c^a = d \ , c^b = d, c^4 = I_3,$$
$$d^a = c \ , d^b = e, d^c = d, d^4 = I_3,$$
$$e^a = e \ \ , e^b = c, e^c = e, e^d = e, e^4 = I_3\}$$

Note that

$$M = < cde > \times < a(cde)^2, b, ce^{-1}, de^{-1} >$$
$$= < cde > \times (M \cap SL(3,5)).$$

3-A generating set for a JS-primitive of $GL(3, p^k)$.

Let F be the field of p^k elements, where $p^k \equiv 1 (\mathrm{mod}\, 3)$. Then since $Sp(2,3)$ is soluble, it follows that there is at just one **JS**-primitive of $GL(3, F)$ whose unique maximal abelian normal subgroup has order p^k-1, namely $M := (C_{p^k-1} \mathrm{Y} E) \ \mathrm{N} \ Sp(2,3)$

Where E is extraspecial of order **27** and exponent **3**.In this section we drive a polycyclic presentation for M, from which we see that $M = C_{p^k-1} \mathrm{Y} (E \succ Sp(2,3))$. This provides a constructive Proof of theorem A in the case $q = 3$. We also need this presentation in the other section when we derive a presentation for one of the **JS**-primitive of $GL(6, p^k)$. We construct a generating set form by the methods described in chapter **2**.Let z be a generator for the scalar group, and define u and v by

and $v := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{bmatrix}, \quad u := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

where ε is a primitive cube root of unity in F .Then $\{u, v, z\}$ generate $Fit(M)$. To extented this set to a generating set for M, we first require a generating set for $Sp(2,3)$ .The set we use consists of the three matrices

$$a\rho := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \ b\rho := \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \ and \ c\rho := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

These matrices satisfy the following relations:

$$(a\rho)^3 = 1$$
$$(b\rho)^{a\rho} = c\rho, (b\rho)^2 = (c\rho)^2$$
$$(c\rho)^{a\rho} = b\rho c\rho, (c\rho)^{b\rho} = (c\rho)^3, (c\rho)^4 = 1.$$

In this case, by the theory in chapter **2** there exist matrices a,b and c of $GL(3,F)$ such that

$$u^a = \lambda_1 uv^2, u^b = \lambda_2 uv, u^c = \lambda_3 u^2 v,$$
$$v^a = \mu_1 v, v^b = \mu_2 uv^2, v^c = \mu_3 uv.$$

Where the λ_i and μ_j are scalars . Setting $\lambda_3 = \mu_3 = \varepsilon^2$ we find that one solution for c is $c := (1-\varepsilon)^{-1}\begin{bmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ 1 & 1 & \varepsilon^2 \end{bmatrix}$.

Then c has determinat **1** and order **4** . Setting $\lambda_2 = 1$ and $\mu_2 = \varepsilon^2$ we find that one solution for b is

$$b := (1-\varepsilon)^{-1}\begin{bmatrix} 1 & \varepsilon & 1 \\ \varepsilon & 1 & 1 \\ \varepsilon & \varepsilon & \varepsilon^2 \end{bmatrix}.$$

Then b has determinat **1**, $b^2 = c^2$ and $c^b = c^3$. Setting $\lambda_1 = \mu_1 = 1$ we find that one solution for a is

$$a := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}.$$

Then a has determinat ε and order 3, and $b^a = c$ and $c^a = bc$. Then we have that $M = < a,b,c,u,v,z >$, where

$$
\begin{aligned}
&a^3 = I_3 \quad ,\\
&b^a = c \quad ,b^2 = c^2,\\
&c^a = bc \quad ,c^b = c^3 \quad ,c^4 = I_3,\\
&u^a = uv^2, u^b = uv \quad ,u^c = \varepsilon^2 u^2 v, u^3 = I_3,\\
&v^a = v \quad ,v^b = \varepsilon^2 uv^2, v^c = \varepsilon uv \quad ,v^u = \varepsilon v \quad ,v^3 = I_3,\\
&z^a = z \quad ,z^b = z \quad ,z^c = z \quad ,z^u = z \quad ,z^v = z \ ,z^{p^{k-1}} = I_3.
\end{aligned}
$$

This yields a polycyclic presentation for M (after replacing ε by $z^{(p^k-1)/3}$).

From this presentation we see that

$$
\begin{aligned}
M &=< z > \mathrm{Y} < a,b,c,u,v >\\
&\cong C_{p^k-1} \mathrm{Y}(E \succ Sp(2,3)).
\end{aligned}
$$

We also have that

$$
M \cap SL(2,F) = \begin{cases} <\varepsilon^{-\frac{1}{3}} a,b,c,u,v> & if \quad p^k \equiv 1 (\mathrm{mod}\, 9),\\ <b,c,u,v> & if \quad p^k \equiv 1 (\mathrm{mod}\, 9). \end{cases}
$$

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