Note On the arts of problem Solving Using The Three SYLOW THEOREMS

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Abstract: We will learn the central roles that the Sylow Theorems play in the theory of finite groups. The arts of problem Solving using the three Slow the theorems are investigated.

Introduction: The Sylow theorems are collections of results in the theory of finite groups. They are a partial converge to Lagrange's Theorem and are one of the most important results in the field. The Sylow Theorems are named for P. Ludwig Sylow, who published their proofs in 1872.

Background Materials: We will use the following definitions on the paper.

Definitions: Let $\boldsymbol{G}$ be a group and $\boldsymbol{p}$ be a prime number:

1) A group of $\boldsymbol{p^k}$ for some $\boldsymbol{k > 0}$ is called a $\boldsymbol{p}$-group.
2) If $\boldsymbol{G}$ is a group of order $\boldsymbol{p^k m}$ where $\boldsymbol{p}$ doesn't divide , then a sub group of $\boldsymbol{p^k}$ is called a Sylow $\boldsymbol{p}$-subgroup of $\boldsymbol{G}$.
3) The number of Sylow $\boldsymbol{p}$-subgroups of $\boldsymbol{G}$ will be denoted by $\boldsymbol{n_p = 1}$ gives a unique Sylow subgroup.

The Main Results We state the three Sylow Theorems without proof and apply them.

Sylow's First Theorem

Every finite group contains a Sylow $\boldsymbol{p}$-subgroup.

Sylow's Second Theorem

In every finite group, the Sylow $\boldsymbol{p}$-subgroups are conjugates.

Sylow's Third Theorem

In every finite group, the number of Sylow $\boldsymbol{p}$-subgroups is equivalent to $\boldsymbol{1}\ mod\ \boldsymbol{p}$ or $\boldsymbol{n_p \equiv 1}\ (mod\ \boldsymbol{p})$

Corollary 1 $\boldsymbol{n_p = 1} \Leftrightarrow \boldsymbol{p}$ is a normal subgroup of .

Corollary 2 Let $\boldsymbol{G}$ be a finite group and let $\boldsymbol{p}$ be a prime.

If $\boldsymbol{p}$ **does not divide** $\boldsymbol{G}$, then the Sylow $\boldsymbol{p}$-subgroups of $\boldsymbol{G}$ is the trivial subgroup.

Corollary 3 A finite abelian group has a unique Sylow $\boldsymbol{p}$-subgroups for each prime $\boldsymbol{p}$.

Theorem 1 Let $\boldsymbol{G}$ be a group such that $\boldsymbol{O(G) = pq}$ with $\boldsymbol{p}$ and $\boldsymbol{q}$ prime $\boldsymbol{p < q}$. If $\boldsymbol{p}$ **does not divide** $\boldsymbol{q - 1}$, then $\boldsymbol{G}$ is cyclic.

Proof Let $\boldsymbol{p}$ be a Sylow $\boldsymbol{p}$-subgroup of $\boldsymbol{G}$ and $\boldsymbol{Q}$ be a Sylow $\boldsymbol{q}$-subgroup of $\boldsymbol{G}$. Since we have

$\boldsymbol{n_p = 1 + kq}$ and $\boldsymbol{n_p | p}$. It follows that $\boldsymbol{k = 0}$. So $\boldsymbol{Q}$ is a normal subgroup of $\boldsymbol{G}$. Now $\boldsymbol{n_p}$ ***does not divide p*** **implies that** either $\boldsymbol{n_p = q}$ or 1. But $\boldsymbol{p}$ **does not divide** $\boldsymbol{q - 1}$ gives us $\boldsymbol{p = 1}$. So $\boldsymbol{p}$ is a normal subgroup of $\boldsymbol{G}$.

Observe that $\boldsymbol{p \cap Q = \{e\}}$ and

$$\boldsymbol{O(pQ) = \frac{O(p) \cdot O(Q)}{O(P \cap Q)} = pq = O(G)}$$

$$\Rightarrow \quad \boldsymbol{G = pQ}$$

$$\Rightarrow \quad \boldsymbol{G} \text{ is cyclic}$$

Corollary 4. Sylow p-subgroups for different primes can only have trivial intersection.
Proof:: If x and y are distinct primes, and P1 is a Sylow- x subgroup of G and P2 is a sylow –y subgroup of G, then
P1 I P2 is a subgroup of both P1 and P2. So
by Lagrange's theorem its order has to divide order of P1 and it also has to divide order of P2, but of course with different primes and x and y the only common factor they have is 1, so P1 I P2 = {e}, the identity element of G.
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Corollary 5 If $\boldsymbol{G}$ is a group and $\boldsymbol{O(G) = 15}$, then $\boldsymbol{G}$ is cyclic.

Proof $\boldsymbol{O(G) = 3 * 5}$ and $\boldsymbol{3}$ **does not divide (5-1)** implies that $\boldsymbol{G}$ is cyclic by Theorem 4.

Corollary 6 If $\boldsymbol{G}$ is a group and $\boldsymbol{O(G) = 35}$, then the Center of $\boldsymbol{G}$, denoted by $\boldsymbol{Z(G)}$, is equal to $\boldsymbol{G}$.

Proof By Theorem 4, $\boldsymbol{G}$ is cyclic and hence is abelian. This implies that $\boldsymbol{Z(G) = G}$.

Example 1. Show that a group of order 200 has a normal Sylow 5-subgroup

Solution:

Easily follows by sylow Third Theorem**.**

Example 2

Let $\boldsymbol{G}$ be a group with $\boldsymbol{O(G) = 99}$, then $\boldsymbol{G}$ is abelian.

Solution $\boldsymbol{O(G) = 3^2 * 11}$. Let $\boldsymbol{H}$ be a Sylow $\boldsymbol{3}$-subgroup of $\boldsymbol{G}$ and $\boldsymbol{K}$ be a Sylow $\boldsymbol{11}$-subgroup of $\boldsymbol{G}$. Applying Sylow third term, we know that both $\boldsymbol{H}$ and $\boldsymbol{K}$ are normal subgroups of $\boldsymbol{G}$ and $\boldsymbol{H \cap K = \{e\}}$.

Now

$$\boldsymbol{O(HK) = \frac{O(H) \cdot O(K)}{O(H \cap K)} = 99 = O(G)}$$

Hence $\boldsymbol{G = HK}$ and is abelian as $\boldsymbol{H}$ and $\boldsymbol{K}$ are abelians.

Example 3 Groups of order $\boldsymbol{340}$ are not simple.

Solution $\boldsymbol{O(G) = 2^2 * 5 * 17}$. Let $\boldsymbol{H}$ be a Sylow $\boldsymbol{5}$-subgroup of $\boldsymbol{G}$. By Sylow third theorem, we have $\boldsymbol{n_5 = 1}$ and hence $\boldsymbol{H}$ is a normal subgroup. Thus by definition of simple groups, $\boldsymbol{G}$ isn't simple.

Example 4 Let $\boldsymbol{G}$ be a group and $\boldsymbol{O(G) = 30}$. Then $\boldsymbol{G}$ isn't simple.

Solution Assume $\boldsymbol{G}$ is simple. Then $\boldsymbol{G}$ has $\boldsymbol{10}$ subgroups of $\boldsymbol{3}$ and $\boldsymbol{6}$ subgroups of order $\boldsymbol{5}$. Note that $\boldsymbol{10}$ subgroups of order $\boldsymbol{3}$ has $\boldsymbol{10(3-1) = 20}$ elements and $\boldsymbol{6}$ subgroups of order $\boldsymbol{5}$ has $\boldsymbol{6(5-1) = 24}$ elements. Hence both have a total number of elements of $\boldsymbol{20 + 24 = 44 > O(G)}$. This is impossible and hence $\boldsymbol{G}$ isn't simple.

Example 5. Let G be a group of order 351. Then G is not simple.

Solution: We have 351= $\boldsymbol{3^3 * 13}$. Note that $\boldsymbol{n_{13}}$ $modulo$ 13 implies that $n_{13} = 1$or 27. If n_{13}=1, then G is not simple as the sylow 13-subgruop of G is a normal subgroup. If n_{13}=27, then we will proceed as follows. Observe that the sylow 13- subgroups are sub groups of order prime, they can only intersect each other at the identity element e. Hence each each sylow 13-subgruops, contains 12 elements of order 13. There are 27 sylow 13 subgroups which implies that the total number elements of order 13 in G to be 27times 12= 324. This gives us that 351-324 = 27 elements of G that don't have order 13. What does this mean? Amazingly this implies that the 27 elements are from a sylow 3 subgroup and hence n_3=1 . Thus, this sylow subgroup is normal and hence G is not simple.

Theorem 2 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be finite groups ad $\boldsymbol{g}$ be a homorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. Let $\boldsymbol{P_1}$ be a Sylow $\boldsymbol{p}$-subgroup of $\boldsymbol{A}$. Then there exists a Sylow $\boldsymbol{p}$-subgroup $\boldsymbol{P_2}$ of $\boldsymbol{B}$ such that $\boldsymbol{g(P_1) \subseteq P_2}$.

Proof $g(P_1)$ is a p-subgroup of B, hence the theorem follows by Sylow's second theorem.

Theorem 3 Let H be a subgroup of G and let p be a Sylow subgroup of H. Then there exists a Sylow p-subgroup A of G such that $p = H \cap K$.

Proof Note that there must be some Sylow p-subgroup K of H. Since $K \cap H$ is a p-subgroup of H, it follows that $p = H \cap K$.

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