e-ISSN 2795-3278
p-ISSN 2795-3274

Check for updates

10.5281/zenodo.7212920 Vol. 05 Issue 10 Oct - 2022 Manuscript ID: #0716

COMPUTATIONAL CONFORMAL GEOMETRY: *A REVIEW*

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ABSTRACT

Conformal geometry is considered a fundamental topic in pure mathematics, including complex analysis, algebraic geometry, Riemann surface theory, differential geometry and algebraic topology. Computational conformal geometry has an important role in digital geometry processing. A good number of practical algorithms are presented to compute conformal mapping, which has been broadly applied in a lot of practical fields, such as computer graphics, wireless sensor networks, medical imaging, visualization, and so on. This work reviews some major concepts and theorems of conformal geometry, their computational methods and the applications for surface parameterization.

KEYWORDS

Conformal Geometry, Ricci flow, Riemannian metric, Euclidean Ricci flow.

1. Introduction

Conformal geometry studies conformal structure of general surfaces. It can handle large deformations and preserves a lot of geometric information during the deformation. In conformal geometry all surfaces can be deformed to three canonical spaces, the sphere, the plane, or the disk. The theoretic base for computational conformal geometry is developing quickly and many practical algorithms converted from classical theories in conformal geometry have been invented. With the advances in mathematical theories and the developed computational technique computational conformal geometry build a bridge between mathematics and computer science. The processing of 3D geometric data with high accuracy is quite challenging. This can be tackled using conformal geometry. In [7] a new method to compute conformal parameterizations are introduced that can parameterize triangular, quadrangular and digital meshes. . J.Wu et al introduce conformal mappings as diffomorphisms to tackle the issue of alignment of deformable tongue images. In [3] a global iterative scheme is adopted to calculate the parameterization coordinates. The parameterization of meshes with disk and toroidal topologies are demonstrated in [6]. This paper is intended to be an overview of concepts and theories related to computational conformal geometry with some applications.

2. Conformal Geometry

The term Conformal refers angle preserving in mathematics. Conformal structure of general surfaces are studied in Conformal Geometry. Conformal structure is a natural geometric structure and a special atlas on surfaces such that angles among tangent vectors can be coherently defined on different local coordinate systems. Conformal map preserves angles. Locally, conformal map introduces only a scale factor to distance and area. Conformal map is intrinsic to the geometry of a mesh, independent of its resolution and preserving the consistency of its orientation.

2.1 Gaussian Curvature

A curved line gradually changes direction from one point to the next. The rate of this change in direction, per unit length along the curve is called the curvature. The maximum and minimum normal curvatures at a point on a surface are defined as the principal (normal) curvatures, and the directions in which these normal curvatures occur are called the principal directions. Gaussian curvature or Gauss curvature K of a surface at a point is the product of the principle curvatures, κ_1 and κ_2, at the given point: $$K = \kappa_1 \kappa_2$$

2.2 Riemann Mapping Theorem

A conformal map between two surfaces preserves angles. Riemann mapping theorem express that any simply connected surface with a single boundary can be conformally mapped to a unit disk. Figure 1 shows a scanned 3D human face, i.e., a topological disk surface denoted asS, mapped to a unit disk denoted as D by a conformal mapping $\varphi : S \rightarrow D$. Suppose γ_1 ,γ_2 are two arbitrary curves on the face surface S, φ maps them to $\varphi(\gamma_1)$,$\varphi(\gamma_2)$. If the intersection angle between γ_1, γ_2 is θ, then the intersection angle between $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ is also θ. Then φ is conformal i.e. angle-preserving.

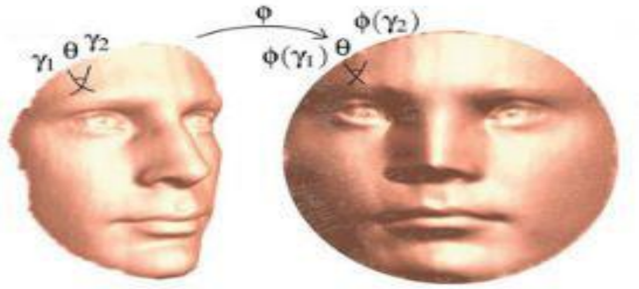

***Figure 1**: A 3D face surface is conformaly mapped to a unit disk. Image from [5]*

2.3 Discrete Metric: Suppose M is a triangular mesh, a discrete metric is a function defined on the edges of M: $l: E \to \mathbb{R}^+$

such that for each triangle $[v_i, v_j, v_k]$, the triangle inequality holds

$$l[v_i, v_j] + l[v_j, v_k] > l[v_k, v_i],\ l[v_j, v_k] + l[v_k, v_i] > l[v_i, v_j],\ l[v_k, v_i] + l[v_i, v_j] > l[v_j, v_k]$$

Let Σ be a function defined on the vertices, $: \Sigma: V \to \mathbb{R}^+$, which assigns a radius γ_i to each vertex v_i. Similarly, let φ be a function defined on the edges, $\varphi: E \to [0, \frac{\pi}{2}]$ which assigns an acute angle $\varphi[v_i, v_j]$ to each edge $[v_i, v_j]$ and is called a weight function on the edges. The pair of vertex radius and edge weight functions, $[M, \Sigma, \varphi]$, is called a circle packing of M.

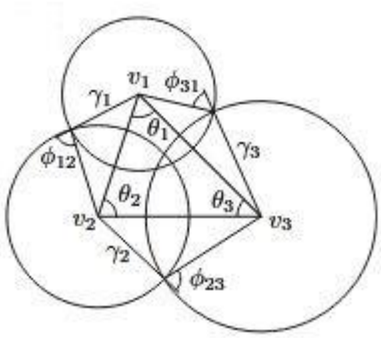

***Figure 2:** Circle packing metric on a triangle. Image from [5]*

Figure 2 visualizes a circle packing. Each vertex v_i has a circle with radius γ_i .For each edge$[v_i, v_j]$, the intersection angle φ_{ij} is defined by the two circles of v_i and v_j , which either intersect or are tangent. If we set a circle packing to a mesh, the circle packing obviously induces a circle packing metric.

2.4 Circle Packing Metric: Suppose M is a triangular mesh, with a circle packing$[M, \Sigma, \varphi]$, then the circle packing metric is defined as

$$l[v_i, v_j] = \sqrt{\gamma_i{}^2 + \gamma_j{}^2 + 2cos\varphi_{ij}\gamma_i\gamma_j}$$

If all the intersection angles are acute, in that case the edge lengths generated by a circle packing satisfy triangle inequality.

If S is a smooth surface with a Riemannian metric $g = g_{ij}$, then a conformal metric is $e^{2u}g$, where $u: S \to \mathbb{R}$ is a function on the surface.[5]

The circle packing approach is well known because of its versatility. It is grounded in theory and more comprehensive in the features of conformal geometry. The mechanics of circle packing are introduced and its connections with conformal geometry are discussed (Monica K. Hurdal and Ken Stephenson, 2004). Circle packing has been used to produce flattened images of cortical surfaces in the sphere, the Euclidean plane, and the hyperbolic plane as well as potential uses of conformal methods in neuroscience and computational anatomy is discussed in [9].

2.5 Conformal Equivalence:

Two Riemannian metrics g,$\hat{g}$ are conformally equivalent if they are related by a positive scaling $\hat{g} = e^{2u}g$ for some real valued function u. On a triangle mesh, the Riemannian metric is captured by the lengths l_{ij} of all edges ij, and two sets of lengths $l, \hat{l}$ are called discretely conformally equivalent if

$$\hat{l}_{ij} = e^{\frac{(u_i+u_j)}{2}} l_{ij}$$

Conformal equivalence offers an appealing strategy for parameterization rather than solve directly for a map to the plane. Conformal maps depending broader geometry processing fails to keep their reliability when the input surface is poorly triangulated or the target curvatures are too extreme. In [8] discrete conformal equivalence of polyhedral surfaces are computed through triangulations. A new algorithm has been developed that provides a novel combination of data structure for tracking correspondence between different triangulations. They also flip the input to an intrinsic Delaunay triangulation to improve the quality of the map. Fig-3 gives an overview of the whole process.

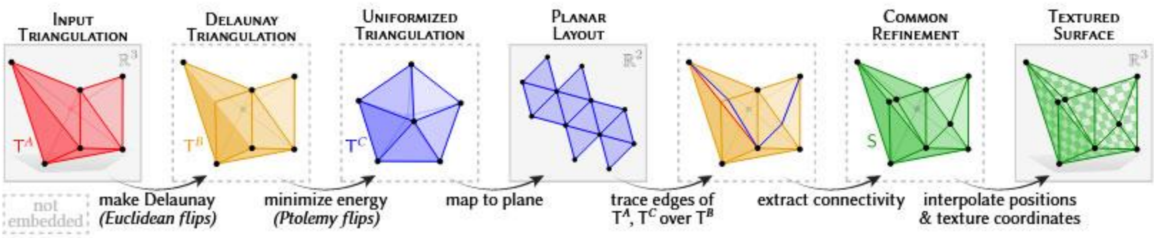

***Figure 3**: Steps of algorithm. Image from [8]*

Definition : (Background Geometry). Suppose R is a discrete metric surface, if each face of R is a spherical, (Euclidean or hyperbolic) triangle, then we say R is with spherical, (Euclidean or hyper-bolic) background geometry. We use S2; E2 and H2to represent spherical Euclidean or hyperbolic back-ground metric.Triangles with different background geometries satisfy different cosine laws. [11]

$$E^2:\ {l_k}^2 = {l_i}^2 + {l_j}^2 - 2l_il_jcos\theta_k$$

$$H^2: cosh{l_k}^2 = coshl_icoshl_j + sinhl_isinhl_jcos\theta_k$$

$$S^2: cos{l_k}^2 = cosl_icosl_j - sinl_isinl_jcos\theta_k$$

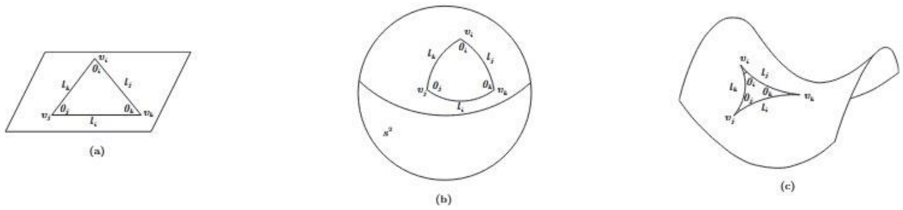

Figure 4: *Different background geometries. (a) Euclidean (b) Spherical (c) Hyperbolic. Image from [10]*

3. Surface Ricci Flow

A surface Ricci flow is the procedure to deform the Riemannian metric of the surface and the deformation is proportional to Gaussian curvatures. Discrete Ricci flow is a dynamic tool for calculating the desired metrics with the prescribed Gaussian curvatures on general surfaces. A lot of applications in graphics and geometric modeling can be formulated as the problem of detecting specific metrics for prescribed curvatures.

Suppose S is a surface embedded in the Euclidean space $\mathbb{R}^3$. It has a Riemannian metric (first fundamental form) induced from the Euclidean metric of $\mathbb{R}^3$, denoted as $\boldsymbol{g} = (g_{ij})_{2\times2}$The Gaussian curvature on interior points and the geodesic curvature on boundary points are calculated by the Riemannian metric g. The total curvature of surface S is determined by the topology of S as described by the Gauss-Bonnet formula,

$$\int_S KdA + \int_{\delta S} k_g dS = 2\pi\chi(S) \tag{1}$$

When δS represents the boundary of S and Kg is the geodesic curvature , $\chi(S)$ is the Euler number of S. The Euler number of a genus g surface with b boundaries is $\chi(S)$ = 2 – 2g – b.

Suppose $u: S \to \mathbb{R}$ is a function defined on the surface S. Then $\bar{g} = e^{2u}g$ is a new metric and it is easy to verify that any angle measure by $\boldsymbol{g}$ equals to that measured by $e^{2u}g$. Therefore, $e^{2u}g$ is said to be conformal to g and e^{2u} is the conformal factor , which measures the area distortion. The Gaussian curvature and geodesic curvature under $\bar{g}$ are

$$\bar{K} = e^{-2u}(-\Delta u + K) \tag{2}$$

$$\overline{K_g} = e^{-u}(\partial_n u + K_g) \tag{3}$$

Where n is the tangent vector perpendicular to the boundary. Suppose the target curvatures $\bar{K}$ and $\overline{K_g}$ are prescribed.

Suppose S is closed surface with a Riemannian metric g. $\bar{K}$ is the target curvature that satisfy equation (1). Then Ricci flow is defined as

$$dg_{ij}/_{dt} = (\bar{K} - K)g_{ij} \tag{4}$$

Preserving the total area $\int_S dA = constant$, where dA(t) is the area element under the metric g(t). [8]

3.1 Discrete Euclidean Ricci Flow

Generally, a surface is represented as a triangular mesh, which is a simplical complex set in $\mathbb{R}^3$. The vertex, edge, and face sets are denoted by V, E, and F, respectively. We denote a vertex as v_i, an edge connecting v_i and v_j as e_{ij}, a face with vertices , v_i, v_j and v_k as f_{ijk}.

The Riemannian metric is approximated by discrete metrics, which are the lengths of edges, $: E \rightarrow \mathbb{R}$. On each triangle f_{ijk}, the edge lengths l_{ij} , l_{jk} and l_{ki} satisfy the triangular inequality. The discrete curvatures are defined as a function on the vertices, $K: V \rightarrow \mathbb{R}$. Suppose v_i is an interior vertex with surrounding faces f_{ijk}, where the corner angles of f_{ijk} at v_i is θ_i^{jk}.Then the discrete Gaussian curvature of v_i is defined as :

$$K_i = 2\pi - \Sigma\theta_i^{jk}; \ \ v_i \notin \ \delta S$$

$$K_i = \pi - \Sigma\theta_i^{jk}; \ \ v_i \in \ \delta S$$

Similar to the smooth surface, discrete curvature also satisfy the Gauss-Bonnet formula,

$$\sum_{v_i \in M/\delta M} K_i + \sum_{v_j \in \delta M} K_j = 2\pi\chi(M)$$

When one mesh is given, the connectivity should be fixed and all possible metrics and the corresponding curvatures should be considered.

Suppose a target curvature $\bar{K}$ for a mesh with circle packing metric (M, Σ, φ) is given. Similar to equation (4) , the discrete Euclidean Ricci flow is defined as

$$d\gamma_i/_{dt} = (\bar{K_i} - K_i)\gamma_i \tag{5}$$

where φ_{ij} is always fixed. Let $u_i = ln\gamma_i$. Then the discrete Euclidean Ricci flow is simply

$$du_i/_{dt} = \bar{K_i} - K_i \tag{6}$$

Which is the negative gradient flow of the function

$$G(u) = -\int_0^u \ \ \ \Sigma_i \overline{K_i} - K_i \ \ )du_i \tag{7}$$

Where $= (u_1, u_2, u_3, \dots \dots \dots, u_n)$. $G(u)$ is called Ricci energy.

The Hassian matrix of $G(u)$

$$\frac{\partial^2 G(u)}{\partial u_i \partial u_j} = \frac{\partial K_i}{\partial u_j} \tag{8}$$

Is positive definite. Therefore Ricci energy has a unique global minimum and Newton's method can be used to minimize it. We can find the desired metric for the prescribed curvature $\bar{K}$ when $G(u)$ is minimum.[5],[11]

3.2 Hyperbolic Ricci Flow

This method is generalized to the hyperbolic background geometry. Suppose M is a mesh, we treat each face as a hyperbolic triangle and each edge as a geodesic . The corner angles and edge lengths are computed using the hyperbolic cosine law.

For a mesh in hyperbolic geometry with circle packing metric, an edge $[v_i, v_j]$ connects two vertices with radii γ_i and γ_j , respectively and the intersection angle between the two circles are φ_{ij} . The edge length $coshl[v_i, v_j] = cosh$ can be computed as

$$coshl[v_i, v_j] = cosh\gamma_i cosh\gamma_j + sinh\gamma_i sinh\gamma_j cos\varphi_{ij}$$

The discrete Gaussian curvature of a vertex is calculated in a similar way as the Euclidean Ricci flow but the discrete conformal factor u_i is defined differently for a hyperbolic or a spherical mesh M with a circle packing metric

$$u_i = \begin{cases} logtanh\frac{\gamma_i}{2} \\ logtan\frac{\gamma_i}{2} \end{cases}$$

The discrete Ricci flow for hyperbolic or spherical mesh is in the same form

$$du_i(t)/_{dt} = \bar{K_i} - K_i$$

The discrete Ricci flow is a negative gradient flow of discrete Ricci energy which is same in discrete Euclidean Ricci flow. To calculate a hyperbolic metric for a surface with negative Euler number, we can simply set the target curvatures of all vertices to zero and then minimize the discrete hyperbolic Ricci energy. Newton's method helps to find the unique global minimum with arbitrary starting point of u.[5]

4. Conclusion

Computational conformal geometry is an emerging field and it has been applied successfully in the computer science field such as graphics, computer vision, wireless sensor network, medical imaging and so on. Though it is successfully used in many fields, there are still many limitations. In future greater theoretic developments and far-reaching applications in computational conformal geometry will be expected by the researchers.

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