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MODELS OF HUMAN POPULATION DYNAMICS

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ABSTRACT

In the study of nature, scientists have devised three models of how two different species may interact. These are the cooperating species model, the competing species model, and the predator-prey model. In this paper, we extend the cooperating species model and the competing species model to the study of human population dynamics.

KEYWORDS

Models, Human Population, Dynamics.

1.0 Introduction

In the study of nature, scientists have devised three models of how two different species may interact. These are the cooperating species model, the competing species model and the predator prey model. In this paper we extend the cooperating species model and the competing species model to the study of human population dynamics.
An example of the predator prey model is the interaction between foxes and rabbits. The foxes are predators that prey on the rabbits as their food source. An example of the cooperating species model is flowering plants and insect pollinators. An example of a competing species model is two fish in a pond that compete for the same food source.

All studies in nature involved only two species. In the study of human population dynamics we are only concerned with cooperating and competing species. Also we use the term population instead of species. In humans there are potentially many, many different populations that can be considered.

Section two presents the current models for two species cooperating and competing species. Section three presents some extended models for cooperating and competing populations. Only the model is presented, no analysis or solutions are proposed.

2.0 Two Species Model

The two species models are due to Lotka and Voltera. Considering two species x and y, the model is stated as

$$dx/dt = a_1x - b_1x^2 + c_1xy$$

$$dy/dt = a_2y - b_2y^2 + c_2xy$$

where the a's represent a population birth parameter, the b's represent a population death parameter and c's represent a population interaction parameter.

The system is cooperative if $c_1 > 0$ and $c_2 >$ o. The system is competitive if $c_1 < 0$ and $c_2 < 0$. This is a system of two differential equations in two unknown and is fairly simple and straightforward.
Note that each equation contains three terms; one for birth, one for death and one for interaction.

3.0 Extended Models

The extended cooperating and competing model are presented. First we consider the case where two populations cooperate against a single competing population in section 3.1. We consider the case of a single population is beset by two competing populations in section 3.2.

3.1 Cooperating Populations

The model of two populations cooperating against a single cooperating population is stated here. Consider three population x, y, z where populations x and y cooperate against population z. The model is:

$$dx/dt = a_1x - b_1x^2 + c_1xy - d_1xz$$
$$dy/dt = a_2y - b_2y^2 + c_2xy - d_2yz$$
$$dz/dt = a_3z - b_3z^2 - c_3xz - d_3yz$$

The first observation is that each equation now consists of four terms; one for population birth rate, one for population death rate and two for the interaction between one population and the other two populations. The system is now a system of three differential equations in three unknowns. This is already more complicated than the two populations models and is not easily solved.
A prime example of this scenario is the war on crime. Under most circumstances the local police population is in competition with the local criminal population. This meets the standard two population model. The two population model is:

$$dP/dt = a_1P - b_1P^2 - c_1PC$$
$$dC/dt = a_2C - b_2C^2 - c_2PC$$

Where P represents the police and C represents the criminals.
In some circumstances local law enforcement teams up with federal law enforcement against criminals. The three population model becomes:

$$dP/dt = a_1P - b_1P^2 + c_1PF - d_1PC$$
$$dF/dt = a_2F - b_2F^2 + c_2PF - d_2FC$$
$$dC/dt = a_3C - b_3C^2 - c_3PC - d_3CF$$

3.1.1 A Demonstration

In this section we demonstrate how two populations can cooperate against a competing population. The example is the war on drugs being carried out by law enforcement.

3.1.1.1 Local Law Enforcement versus Drugs

Consider the mathematical model

$$f_D = a_1P/(1+d_3P) - a_{NR}DP/(1+d_2P) - b_1D^2 = 0 = f_D(D, P) \quad (1)$$
$$f_P = a_2P/(1+d_3P) - a_{NR}DP/(1+d_2D) - b_2P^2 = 0 = f_P(D, P) \quad (2)$$

The populations D(t) and P(t) represent the populations of the drug users and police populations. New drug users are slowly coming into the drug user population. The parameters are all assumed to be positive and their descriptions are given in Table 1.

Table 1: List of parameters used in the differential equation model

Symbols	Meaning
a_1	Growth rate of the drug population
a_2	Growth rate of the police population
b_1	Population loss in D due to intra-species competition and natural mortality
b_2	Population loss in P due to intra-species competition and natural mortality
a_{NR}	Maximum per capita loss in D due to recruitment by drug users
d_1	Measures the effectiveness of P in disrupting the growth rate of D
d_2	Measures the resilience of P to recruitment strategies by D
d_3	Measures the effectiveness of P in the growth of drug users

In the case of $d_i = b_i = 0$, the mathematical model becomes similar to the competing species model. The parameters d_i influence the carrying capacity of the individual populations. Or instance, if $d_1 >> 1$ then the growth rate of D is reduced. This is interpreted as: a highly effective radicalized population can greatly hinder the growth rate of D. The growth rate of the drug population depends on the successful recruitment from the neutral population. Notice, that if $d_2 >> 1$ then the recruitment by P is small, Also, if $d_3 >> 1$, new drug members are introduced into the drug population at a slower rate. The values chosen for the variables in this model are listed in Table 2.

Table2: Values of parameters

a_1	a_2	b_1	b_2	a_{NR}	d_1	d_2	d_3
2	2	0.5	0.5	2	2	2	3

3.1.1.2 Police Drug (P, D) ODE Model

Consider the mathematical model

$$f_D(D, P) = (a_1/(1+d_3D) – a_{NR}D/(1+d_2P) – b_1P) P = 0 \quad (3)$$

$$f_P(D, P) = (a_2/(1+d_3P)) - a_{NR}P/(1+d_2P) –(b_2D)) D= 0 \quad (4)$$

Since this system is nonlinear, the first step is linearization using the Jacobian.
The Jacobian for this system is defined as

$$J = \begin{vmatrix} \partial f_p/\partial D & \partial f_p/\partial P \\ \partial f_D/\partial D & \partial f_D/\partial P \end{vmatrix}$$

The partial derivatives are:,

$$\partial P/\partial P = a_1/(1+d_1D) – a_{nr}D(1+d_2P) – a_{nr}d_2DP – 2b_1P$$
$$\partial P/\partial D = -a_1d_1P/(1+d_1D)\hat{}2 – a_{nr}P/(1+d_2P)$$
$$\partial D/\partial P = -a_2d_3D/(1+d_1D)\hat{}2 –a_{nr}P/(1+d_2P)$$
$$\partial D/\partial D = a_2/(1+d_3P) - a_{nr}D(1+d_2P) – an_rd_2DP – 2b_2D$$

Using the values in table for the parameters, the Jacobian becomes.

$$J = \begin{vmatrix} 2/(1+d_1D) – 2D(1+2P) – 4DN – P & P/(1+2D)\hat{}2 – 2P/(1+2P) \\ -4D/(1+2D)\hat{}2 –2P/(1+2P) & 2/(1+2P) - 2_rD(1+2P) – 4DP –D \end{vmatrix}$$

3.1.2 Local Law Enforcement Plus Federal Law Enforcement

In this section we look at local law enforcement teaming up with federal law enforcement against drug populations. Consider the mathematical model:

$$f_G(G, P, D) = a_1G/(1+d_3G) + a_{nr}GP/(1+d_2G) – a_{nr}GD/(1+d_2G) – b_1G^2$$
$$f_P(G, P, D) = a_1P/(1+d_3P) + a_{nr}GP/(1+d_2P) – a_{nr}PD/(1+d_2P) – b_1P^2$$
$$f_D(G, P, D) = a_2D/(1+d_3D) – a_{nr}DG/(1+d_2D) – a_{nr}DP/(1+d_2D) – b_2D^2$$

Since the system is non-linear, the first step is to linearize using the Jacobian where the Jacobian is defined as:

$$J = \begin{vmatrix} \partial f_G/\partial G & \partial f_G/\partial P & \partial f_G/\partial D \\ \partial f_P/\partial G & \partial f_P/\partial P & \partial f_P/\partial D \\ \partial f_D/\partial G & \partial f_D/\partial P & \partial f_D/\partial D \end{vmatrix}$$

The partial derivatives are:

$$\partial f_G/\partial G = [a_1d_3G – a_1(1+d_3G)]/(1+d_3G)^2 + [a_{nr}d_2GP – a_{nr}P(1+d_2G)]/(1+d_2G)^2 –$$
$$[a_{nr}d_2GD – a_{nr}D(1+d_2G)]/(1+d_2G)^2 – 2b_1G$$

$$\partial f_G/\partial P = a_{nr}G/(1+d_2G)$$

$$\partial f_G/\partial D = - a_{nr}G/(1+d_2G)$$

$$\partial f_P/\partial G = a_{nr}P/(1+d_2P)$$

$$\partial f_P/\partial P = [a_1d_3P – a_1(1+d_3P)]/(1+d_3P)^2 + [a_{nr}d_2GP – a_{nr}G(1+d_2P)]/(1+d_2P)_2 –$$
$$[a_{nr}d_2PD – a_{nr}D(1+d_2P)]/(1+d_2P)^2 – 2b_1P$$

$$\partial f_P/\partial D = - a_{nr}P/(1+d_2P)$$

$\partial f_D/\partial G = - a_{nr}D/(1+d_2D)$

$\partial f_D/\partial P = - a_{nr}D/(1+d_2D)$

$\partial f_D/\partial D = [\, a_2d_3D - a_2(1+d_3D)]/(1+d_3D)^2 - [\, a_{nr}d_2DG - a_{nr}G(1+d_2D)]/(1+d_2D)^2$

$- \quad - [\, a_{nr}d^2DP - a_{nr}P(1+d_2D)]/(1+d_2D)^2 - 2b_2D$

These partial derivatives can be plugged back into the Jacobian:

| $[a_1d_3G - a_1(1+d_3G)]/(1+d_3G)^2 + [a_{nr}d_2GP - a_{nr}P(1+d_2G)]/(1+d_2G)^2 -$

$[a_{nr}d_2GD - a_{nr}D(1+d_2G)]/(1+d_2G)^2 - 2b_1G$

$a_{nr}G/(1+d_2G)$ |

| $- a_{nr}G/(1+d_2G)$ |

| |

J = | $a_{nr}P/(1+d_2P)$

$[\, a_1d_3P - a_1(1+d_3P)]/(1+d_3P)^2 + [\, a_{nr}d_2GP -$

$a_{nr}G(1+d_2P)]/(1+d_2P)_2 -$

$[\, a_{nr}d_2PD - a_{nr}D(1+d_2P)]/(1+d_2P)^2 - 2b_1P$ |

| $- a_{nr}P/(1+d_2P)$ |

| |

| $- a_{nr}D/(1+d_2D)$ |

| $- a_{nr}D/(1+d_2D)$ |

| $[\, a_2d_3D - a_2(1+d_3D)]/(1+d_3D)^2$

$- [\, a_{nr}d_2DG - a_{nr}G(1+d_2D)]/(1+d_2D)^2$

$- [\, a_{nr}d^2DP - a_{nr}P(1+d_2D)]/(1+d_2D)^2$
$- 2b_2D$ |

At this point it would be desirable to compare the stability of the system of two population model to the stability of the three population model. Unfortunately the three population model is non-trivial so such a comparison is not feasible.

3.2 Competing Populations

The model of two populations competing with one other population is stated here. Consider the populations x, y, z where populations y and z are in competition with x. The model is

$$dx/dt = a_1x - b_1x^2 - c_1xy - d_1xz$$

$$dy/dt = a_2y - b_2y^2 - c_2xy - d_2yz$$

$$dz/dt = a_3z - b_3z^2 - c_3xz - d_3yz$$

This is another system of three equations in three unknowns.

A prime example of this is also the war on crime. Police typically have a gang task force to deal with gang populations and a drug task force to deal with drug populations. This fits the three population model of one population facing two competing populations. The three population model becomes:

$$dP/dt = a_1P - b_1P^2 - c_1PD - d_1PG$$

$$dD/dt = a_2D - b_2D^2 - c_2PD - d_2DG$$

$$dG/dt = a_3G - b_3G^2 - c_3PG - d_3DG$$

3.2.1 A Demonstration

In this section we demonstrate how two populations can compete against a population. The example is the war on drugs being carried out by law enforcement.

3.2.1.1 Local Law Enforcement Versus Drugs

Consider the mathematical model

$$f_P(D, P)= a_1P(1+d_1P) - a_{NR}DP/(1+d_2P) - b_1D^2 \quad (1)$$

$$f_D(D, P)= a_2P/(1+d_3P) - a_{NR}DP/(1+d_2D) - b_2P^2 \quad (2)$$

The populations D(t) and P(t) represent the populations of the drug users and police populations. New drug users are slowly coming into the drug user population. The parameters are all assumed to be positive and their descriptions are given in Table 3.

Table 3: List of parameters used in the differential equation model

Symbols	Meaning
a_1	Growth rate of the drug population
a_2	Growth rate of the police population
b_1	Population loss in D due to intra-species competition and natural mortality
b_2	Population loss in P due to intra-species competition and natural mortality
a_{NR}	Maximum per capita loss in D due to recruitment by drug users
d_1	Measures the effectiveness of P in disrupting the growth rate of D
d_2	Measures the resilience of P to recruitment strategies by D
d_3	Measures the effectiveness of P in the growth of drug users

In the case of $d_i = b_i = 0$, the mathematical model becomes similar to the competing species model. The parameters d_i influence the carrying capacity of the individual populations. Or instance, if d_1>> 1 then the growth rate of D is reduced. This is interpreted as: a highly effective radicalized population, can greatly hinder the growth rate of D. The growth rate of the drug population depends on the successful recruitment from the neutral population. Notice, that if d_2>> 1 then the recruitment by P is small, Also, if d_3>> 1, new drug members are introduced into the drug population at a slower rate. The values chosen for the variables in this model are listed in Table4.

Table 4: Values of parameters

a_1	a_2	b_1	b_2	a_{NR}	d_1	d_2	d_3
2	2	0.5	0.5	2	2	2	3

3.2.1.2 Police Drug (P, D) ODE Model

Consider the mathematical model

$$f_D(D, P) = (a_1/(1+d_3D) – a_{NR}D/(1+d_2P) – b_1P) P = 0 \quad (3)$$

$$f_P(D, P) = (a_2/(1+d_1P)) - a_{NR}P/(1+d_2P) –(b_2D)) D= 0 \quad (4)$$

Since this system is nonlinear, the first step is linearization using the Jacobian.
The Jacobian for this system is defined as

$$J = \begin{vmatrix} \partial f_p/\partial D & \partial f_p/\partial P \\ \partial f_D/\partial D & \partial f_D/\partial P \end{vmatrix}$$

The partial derivatives are:,

$$\partial P/\partial P = a_1/(1+d_1D) – a_{nr}D(1+d_2P) – a_{nr}d_2DP – 2b_1P$$

$$\partial P/\partial D = -a_1d_1P/(1+d_1D)\text{^}2 – a_{nr}P/(1+d_2P)$$

$$\partial D/\partial P = -a_2d_3D/(1+d_1D)\text{^}2 –a_{nr}P/(1+d_2P)$$

$$\partial D/\partial D = a_2/(1+d_3P) - a_{nr}D(1+d_2P) – an_rd_2DP – 2b_2D$$

Using the values in table for the parameters, the Jacobian becomes.

$$J = \begin{vmatrix} 2/(1+d_1D) – 2D(1+2P) – 4DN – P & P/(1+2D)\text{^}2 – 2P/(1+2P) \\ -4D/(1+2D)\text{^}2 –2P/(1+2P) & 2/(1+2P) - 2_rD(1+2P) – 4DP –D \end{vmatrix}$$

3.2.1.5 Local Law Enforcement Versus Drugs and Gangs

In this section we show one population being opposed by two competing populations. We continue with the law enforcement example this time pitting a law enforcement population against both a drug population and a gang population. Consider the mathematical model:

$$f_P(P, D, G) = a_1P/(1+d_3P) – a_{nr}PD/(1+d_2P) – a_{nr}PG/(1+d_2P) – b_1P^2$$

$$f_D(P, D, G) = a_2D/(1+d_3D) – a_{nr}PD/(1+d_2D) +a_{nr}DG/(1+d_2D) – b_2D^2$$

$$f_G(P, D, G) = a_2G/(1+d_3G) – a_{nr}PG/(1+d_2G) + a_{nr}GD/(1+d_2G) – b_2G^2$$

Since the system is non-linear the first step is to linearize with the Jacobian where the Jacobian is defined as:

$$J = \begin{vmatrix} \partial f_P/\partial P & \partial f_P/\partial D & \partial f_P/\partial G \\ \partial f_D/\partial P & \partial f_D/\partial D & \partial f_D/\partial G \\ \partial f_G/\partial P & \partial f_G/\partial D & \partial f_G/\partial G \end{vmatrix}$$

The partial derivatives are:

$$\partial f_P/\partial P = [a_1d_3P – a_1(1+d_3P)]/(1+d_3P)2 -] a_{nr}d_2PG – a_{nr}D(1+d_2P)]/(1+d_2P)^2$$
$$- [a_{nr}d_2PG – a_{nr}G(1+d_2P)]/(1+d_2P)^2 – 2b_1P$$

$$\partial f_P/\partial D = - a_{nr}P/(1+d_2P)$$

$$\partial f_P/\partial G = - a_{nr}P/(1+d_2P)$$

$$\partial f_D/\partial P = - a_{nr}D/(1+d_2D)$$

$$\partial f_D/\partial D = [a_2d_3D – a_2(1+d_2D)]/(1+d_3D)^2 – [a_{nr}d_2PD – a_{nr}P(1+d_2D)]/(1+d_2D)^2 +$$

$[\, a_{nr}d_2DG - a_{nr}(1+d_2D)]/1+d_2D)^2 - 2b_2D$

$\partial f_D/\partial G = a_{nr}D/(1+d_2D)$

$\partial f_G/\partial P = -\, a_{nr}G/(1+d_2G)$

$\partial f_G/\partial D = a_{nr}G/(1+d_2G)$

$\partial f_G/\partial G = [\, a_2d_3G - a_2(1+d_3G)]/(1+d_3G)^2 - [\, a_{nr}d^2PG - a_{nr}P(1+d_2G)]/(1+d_2G)^2 +$

$[\, a_{nr}d_2GD - a_{nr}D(1+d_2G)]/(1+d_2G)2 - 2b_2G$

The partial derivatives can then be plugged back into the Jacobian

$|\ [\, a_1d_3P - a_1(1+d_3P)]/(1+d_3P)2 -]\, a_{nr}d_2PG - a_{nr}D(1+d_2P)]/(1+d_2P)^2$

$-\ [\, a_{nr}d_2PG - a_{nr}G(1+d_2P)]/(1+d_2P)^2 - 2b_1P$

$-\ a_{nr}P/(1+d_2P)$

$-\ a_{nr}P/(1+d_2P)$ |

| |

J = | = $-\ a_{nr}D/(1+d_2D)$

$[\, a_2d_3D - a_2(1+d_2D)]/(1+d_3D)^2 - [a_{nr}d_2PD -$

$a_{nr}P(1+d_2D)]/(1+d_2D)^2 +$

$[\, a_{nr}d_2DG - a_{nr}(1+d_2D)]/1+d_2D)^2 - 2b_2D$ |

$a_{nr}D/(1+d_2D)$ |

| |

| $-a_{nr}G/(1+d_2G)$

$a_{nr}G/(1+d_2G)$

$[\, a_2d_3G - a_2(1+d_3G)]/(1+d_3G)^2$

$-\ [\, a_{nr}d^2PG - a_{nr}P(1+d_2G)]/(1+d_2G)^2 +$

$[\, a_{nr}d_2GD - a_{nr}D(1+d_2G)]/(1+d_2G)2$

$-\ 2b_2G$ |

Once again it would be desirable to compare the stability of the system of two population model to the stability of the three population model. Unfortunately the three population model is non-trivial so such a comparison is not feasible.

3.3 A Larger Example

The war on crime also presents the possibility of an even more complex system. This is the scenario when local law enforcement teams up with federal law enforcement against both gang and drug populations. The four population model presents as:

$$dP/dt = a_1P - b_1P^2 - c_1PG - d_1PD + e_1PF$$
$$dF/dt = a_2F - b_2F^2 - c_2FG - d_2FD + e_2PF$$
$$dG/dt = a_3G - b_3G^2 - c_3PG - d_3GF + e_3GD$$
$$dD/dt = a_2D - b_4D^2 - c_4PD - d_4FD + e_4GD$$

This is a system of four differential equations in four unknowns which is even more complicated than the three population model.

3.4 Expanding the Models Further

It should be clear that the model from section 3.1can easily be expanded to three cooperating populations and two competing populations.

$$du/dt = a_1u – b_1u^2 + c_1uv + d_1ux – e_1uy – f_1uz$$
$$dv/dt = a_2v –b_2v^2 +c_2uv + d_2vx – e_2vy – f_2vz$$
$$dx/dt = a_3x – b_3x^2 +c_3ux + d_3vx – e_3xy – f_3xz$$
$$dy/dt = a_4y – b_4y^2 – c_4uy – d_4vy – e_4xy – f_4yz$$
$$dz/dt = a_5z – b_5z^2 – c_5uz – d_5vy-e_5xz – f_5yz$$

It should also be clear that the model from section 3.2 can easily be expanded to two cooperating populations and three competing populations.

$$du/dt = a_1u – b_1u^2 + c_1uv - d_1ux – e_1uy – f_1uz$$
$$dv/dt = a_2v – b_2v^2 + c_2uv - d_2vx – e_2vy – f_2vz$$
$$dx/dt = a_3x – b_3x^2 - c_3ux - d_3vx – e_3xy – f_3xz$$
$$dy/dt = a_4y – b_4y^2 – c_4uy – d_4vy – e_4xy – f_4yz$$
$$dz/dt = a_5z – b_5z^2 – c_5uz – d_5vy-e_5xz – f_5yz$$

Further extensions should be obvious.
One observation is in order. Each time a new population is introduced into the model another term is added to each equation. Complexity grows quickly.

4.0 Conclusions

This paper has extended the two species cooperating species model and the two species competing species model for the purpose of studying human population dynamics. One model was presented where two populations cooperate against a competing population. Another model was presented where one population was beset by two competing populations. Both of these models consisted of a system of three differential equations in three unknowns. These three population models are already more complex than the two species models.
Also presented was a four population model where two populations cooperated against two other competing populations. This system is even more complex than the three population models.
Since there are potentially many different human populations that can interact in very many different ways, the study of human population dynamics promises to be both very challenging and very exciting.

5. References

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