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NOTE ON THE ATTRACTIVE MATHEMATICAL PATTERNS OF THE MULATU NUMBERS

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ABSTRACT

The Mulatu numbers were introduced by Mulatu Lemma in [1].The Mulatu numbers are integral sequences of numbers of the form: **4, 1, 5,6,11,17,28,45...***These numbers have wonderful and amazing properties and patterns.*

In mathematical terms, the sequence of the Mulatu numbers is defined by the following recurrence relation:

$$M_n := \begin{cases} 4 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ M_{n-1} + M_{n-2} & \text{if } n > 1. \end{cases}$$

The first number of the sequence is 4, the second number is 1, and each subsequent number is equal to the sum of the previous two numbers of the sequence itself. That is, after two starting values, each number is the sum of the two preceding numbers. In [1] some properties and patterns of the numbers were considered. In this paper, we more deeply examine additional properties and patterns of these fascinating and mysterious numbers. Many beautiful mathematical identities involving the Mulatu numbers, the Fibonacci numbers and the Lucas numbers will be explored.

2000Mathematical Subject Classification: 11

KEYWORDS

Mulatu numbers, Mulatu sequences, Fibonacci numbers, Lucas numbers, Fibonacci sequences, and Lucas sequences.

1. **Introduction and Background**. As given in [1], the Mulatu numbers are a sequence of numbers recently introduced by Mulatu Lemma, an Ethiopian Mathematician and Distinguished Professor of Mathematics at Savannah State University, Savannah, Georgia, and USA. The Mulatu sequence has wealthy mathematical properties and patterns like the two celebrity sequences of Fibonacci and Lucas.

In this paper, more interesting relationships of the Mulatu numbers to the Fibonacci and Lucas numbers will be presented.

Here are the First 21 Mulatu, Fibonacci, and Lucas numbers for quick reference.

Mulatu(M_n)**,Fibonacci**(F_n)**and** **Lucas**(L_n)**Numbers**

(Tables 1 &2)

Table 1

n:	0	1	2	3	4	5	6	7	8	9	10	11
M_n:	4	1	5	6	11	17	28	45	73	118	191	309
F_n:	0	1	1	2	3	5	8	13	21	34	55	89
L_n:	2	1	3	4	7	11	18	29	47	76	123	199

Table 2

n:	12	13	14	15	16	17	18	19	20
M_n	500	809	1309	2118	3427	5545	8972	14517	23489
F_n:	144	233	377	610	987	1597	2584	4181	6765
L_n:	322	521	843	1364	2207	3571	5778	9349	15127

Remark1: Throughout this paper M, F, and L stand for Mulatu numbers, Fibonacci numbers, and Lucas number respectively.

The following well-known identities of Mulatu numbers [1], Fibonacci numbers, and Lucas numbers are required in this paper and hereby listed for quick reference.

(1) $L_n = F_{n-1} + F_{n+1}$

(2) $F_{n+1} = F_n + F_{n-1}$

(3) $M_n = L_n + 2F_{n-1}$.

(4) $F_{2n} = F_n L_n$

(5) $5F^2_n - L^{2_n} = 4(-1)^{n+1}$

(6) $F_n = \dfrac{L_{n+1} + L_{n-1}}{5}$

(7) $L_{n+1} = L_n + L_{n-1}$

(8) $F_{n+k} = F_{n-1}F_k + F_nF_{k+1}$

(9) $M_{-n} = (-1)^n M_n$

(10) $L_{n+m} = \dfrac{5F_nF_m + L_nL_m}{2}$

<u>The Main Results</u>.

Proposition 1.

$$M^2_n - M^2_{n+1} - M^2_{n-1} + 2M_{n+1}M_{n-1} = 0$$

Proof: The proposition easily follows using the recurrence formula

$$M_{n+1} = M_n + M_{n-1}$$

Proposition 2.

$$M_n = 4F_{n-1} + F_n$$

Proof: Theorem 9 [1] implies that $F_{n-1} = \dfrac{M_n - F_{n+1}}{3}$. Thus we have

$$3F_{n-1} + F_{n+1} = M_n \Rightarrow M_n = 4F_{n-1} + F_n$$

Theorem 1.

$$M_{n+2} = 7F_{n+1} - L_n$$

Proof: Note that from above $F_n = \dfrac{L_{n+1} + L_{n-1}}{5}$

$$\Rightarrow 5F_{n-1} = L_n + L_{n-2} = (M_n - 2F_{n-1}) + L_{n-2}$$

$$\Rightarrow 7F_{n-1} - L_{n-2} = M_n$$

$$\Rightarrow M_{n+2} = 7F_{n+1} - L_n$$

Theorem 2.

(a) If M_n is divisible by 2, then $M^2_{n+1} - M^2_{n-1}$ is divisible by 4

(b) If M_n is divisible by 3, then $M^3_{n+1} - M^3_{n-1}$ is divisible by 9.

Proof: Note that:

(a) $M^2_{n+1} - M^2_{n-1}$

$$= (M_{n+1} - M_{n-1})(M_{n+1} + M_{n-1}) = M_n(M_n + M_{n-1} + M_{n-1}) = M^2_n + 2M_nM_{n-1}.$$

Now it is easy to see that if M_n is divisible by 2, then $M^2_{n+1} - M^2_{n-1}$ is divisible by 4

(b) $M^3{}_{n+1} - M^3{}_{n-1} = (M_{n+1} - M_{n-1})(M^2{}_{n+1} + M_n M_{n-1} + M^2{}_{n-1})$

$= M_n\ (M^2{}_{n+1} + M_{n+1}M_{n-1} + M^2{}_{n-1})$

$= M_n((M_n + M_{n-1})^2 + M_{n-1}(M_n + M_{n-1}) + M^2{}_{n-1})$

$= M_n(M^2{}_n + 3M_n M_{n-1} + 3M^2{}_{n-1})$

$= M^3{}_n + 3M^2{}_n M_{n-1} + 3M_n M^3{}_{n-1}$

Hence M_n is divisible by 3 $\Rightarrow M^3{}_{n+1}$ - $M^3{}_{n-1}$ is divisible by 9.

Theorem 3.The addition formula *for* **Mulatu** numbers.

$$M_{n+k} = F_{n-1}M_k + F_n M_{k+1}$$

Proof: By Theorem 8[1] we have,

$$M_n = F_{n-3} + F_{n-1} + F_{n+2}\,.$$

Hence it follows that

$$M_{n+k} = F_{n+k-3} + F_{n+k-1} + M_{n+k+2}\,.$$

Now using the addition formula for **Fibonacci** numbers given above, it follows that

$$M_{n+k} = (F_{n-1}F_{k-3} + F_n\ F_{k-2}) + (F_{n-1}F_{k-1} + F_n F_k) + (F_{n-1}F_{k+2} + F_n F_{k+3})$$

$$= (F_{n-1}F_{k-3} + F_{n-1} + \mathrm{F}_{k-1} + F_{n-1}F_{k+2}) + (\mathrm{F}_n\ \mathrm{F}_{k-2} + \mathrm{F}_n\ F_k + F_n F_{k+3})$$

$$= F_{n-1}(F_{k-3} + \mathrm{F}_{k-1} + F_{k+2}) + F_n(\mathrm{F}_{k-2} + F_k +{}_n F_{k+3})$$

$$= F_{n-1}\ M_k + F_n\ M_{k+1}\,.$$

Hence the theorem is proved.

Theorem 4:

$$M_{2n-1} = F_{2n} - 3F^2{}_{n-1} + 6F_n F_{n-1}$$

Proof: By Theorem 3 we have,

$$M_{2n-1} = M_{n+(n-1)} = F_{n-1}M_{n-1} + F_n M_n$$

$$= F_{n-1}M_{n-1} + F_n(L_n + 2F_{n-1})$$

$$= F_{n-1}M_{n-1} + F_n L_n + 2F_n F_{n-1}$$

$$= F_{n-1}M_{n-1} + F_{2n} + 2F_n F_{n-1}\,.$$

Now applying *Theorem 3*to M_{n-1}, we have

$$M_{n-1} = M_{(n-1)+0} = F_{n-2}M_0 + F_{n-1}M_1 = 4F_{n-2} + F_{n-1} \text{ and}$$

$$4F_{n-2} + F_{n-1} = 4(F_n - F_{n-1}) + F_{n-1} = -3F_{n-1} + 4F_n.$$

Hence, $M_{2n-1} = F_{2n} + F_{n-1}(-3F_{n-1} + 4F_n) + 2F_n F_{n-1} = F_{2n} - 3F^2{}_{n-1} + 6F_n F_{n-1}$

Theorem4. The Subtraction formula for **Mulatu** numbers

$$M_{n-k} = 4F_{n-k+1} - 3F_{n-k}$$

Proof: $M_{n-k} = M_{(n=k)+0}$ and hence by Theorem 3, we have

$$M_{n-k} = F_{n-k-1}M_0 + F_{n-k}M_1$$

$$= 4F_{n-k-1} + F_{n-k}$$

$$= 4(F_{n-k-1} + F_{n-k}) - 3F_{n-k}$$

$$= 4F_{n-k+1} - 3F_{n-k}.$$

Corollary 1.

$$M_n = 4F_{n+1} - 3F_n.$$

Theorem 5.

$$F_{2n} - M_n F_{n+1} - F_{n+1}F_n = -L^2_n$$

Proof: We use the identities listed above to prove the theorem.

Note that $F_{2n} - M_n F_{n+1} - F_{n+1}F_n = F_nL_n - M_n F_{n+1} - F_{n+1}F_n$

$$= F_n(F_{n-1} + F_{n+1}) - F_{n+1}(L_n + 2F_{n-1}) - F_{n+1}F_n$$

$$= F_n(F_{n-1} + F_{n+1}) - (F_n + F_{n-1})(L_n + 2F_{n-1}) - F_{n+1}F_n$$

$$= F_n(F_{n-1} + F_{n+1}) - (F_n + F_{n-1})(F_{n+1} + F_{n-1} + 2F_{n-1}) - (F_n + F_{n-1})F_n$$

$$= F_n(F_{n-1} + F_n + F_{n-1}) - (F_n + F_{n-1})(F_n + F_{n-1} + F_{n-1} + 2F_{n-1}) - (F_n + F_{n-1})F_n$$

$$= F_n(2F_{n-1} + F_n) - (F_n + F_{n-1})(F_n + 4F_{n-1}) - (F_n + F_{n-1})F_n$$

$$= 2F_nF_{n-1} + F^2_n - F^2_n - 4F_nF_{n-1} - F_nF_{n-1} - 4F_{n-1}^2 - F^2_n - F_{n-1}F_n$$

$$= -F^2_n - 4F_nF_{n-1} - 4F^2_{n-1}$$

$$= -(F^2_n + 4F_nF_{n-1} + 4F^2_{n-1})$$

$$= -(F_n + 2F_{n-1})^2$$

$$= -(F_n + F_{n-1} + F_{n-1})^2$$

$$= -(F_{n+1} + F_{n-1})^2$$

$$= -L^2_n$$

The following result deals with the half-angle type formula. It is rather an amazingly interesting strong result.

Theorem 6. Fundamental identity.

$$M_{2n} = M_nL_n + 4(-1)^{n+1}$$

Proof: By *Theorem 3*, $M_{2n} = M_{n+n} = F_{n-1} M_n + F_nM_{n+1}$. Again applying Theorem 3, to M_{n+1} and using $L_n = F_{n+1} + F_{n-1}$, we get

$$M_{2n} = F_{n-1} M_n + F_n(F_{n-1}M_1 + F_nM_2)$$

$$= F_{n-1} M_n + F_n(F_{n-1} + 5F_n).$$

$$= F_{n-1} M_n + F_nF_{n-1} + 5F^2_n.$$

$$=((L_n - F_{n+1})M_n + F_nF_{n-1} + 5F^2_n$$

$$= L_nM_n - F_{n+1} M_n + F_nF_{n-1} + 5F^2{}_n$$
$$= L_nM_n - (F_n + F_{n-1})(L_n + 2F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$
$$= L_nM_n - (F_n + F_{n-1})(F_{n+1} + F_{n-1} + 2F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$
$$= L_nM_n - (F_n + F_{n-1})(F_n + 4F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$
$$= L_nM_n - F^2{}_n - 4F_nF_{n-1} - F_nF_{n-1} - 4F^2{}_{n-1} + F_nF_{n-1} + 5F^2{}_n$$
$$= L_nM_n - F^2{}_n - 4F_nF_{n-1} - 4F^2{}_{n-1} + 5F^2{}_n$$
$$= L_nM_n - (F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1}) + 5F^2{}_n$$

From the proof of *Theorem 5*, we know that $F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1} = L^2{}_n$.

Hence $M_{2n} = L_nM_n - L^2{}_n + 5F^2{}_n$. Now using that $5F^2{}_n - L^2{}_n = 4(-1)^{n+1}$

it easily follows that $M_{2n} = L_nM_n + 4(-1)^{n+1}$.

Remark 2: Note that using Corollary 1, we can also express M_{2n} as follows:

$$M_{2n} = 4F_{2n+1} - 3F_{2n}.$$

Corollary 2.

$$M_{2n} = L^2{}_n + 4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1}$$

Proof: We have $M_{2n} = M_nL_n + 4(-1)^{n+1}$

$$= (L_n + 2F_{n-1})L_n + 4(-1)^{n+1}$$
$$= L^2{}_n + 2F_{n-1}L_n + 4(-1)^{n+1}$$
$$= L^2{}_n + 2F_{n-1}(F_{n+1} + F_{n-1}) + 4(-1)^{n+1}$$
$$= L^2{}_n + 2F_{n-1}(F_n + F_{n-1} + F_{n-1}) + 4(-1)^{n+1}$$
$$= L^2{}_n + 4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1}$$

Corollary 3.Square Expansion

$$M^2{}_n = M_{2n} + 2M_nF_{n-1} + 4(-1)^n$$

Proof: Note that

$$M^2{}_n = M_nM_n = M_n(L_n + 2F_{n-1}) = M_nL_n + 2M_nF_{n-1}.$$

Hence the corollary follows by *Theorem6*.

Theorem 7.

$$\frac{9F_n^{\ 2} + L^2{}_2 + 4F^2{}_{n-1}}{2} = L_nM_n + 4(-1)^{n+1}$$

Proof: $\dfrac{9F_n^{\ 2} + L^2{}_n + 4F^2{}_{n-1}}{2} = \dfrac{5F_n^{\ 2} + L^2{}_n + 4F_n^{\ 2} + 4F^2{}_{n-1}}{2}$

$$= \frac{5F_n^{\ 2} + L_n^{\ 2}}{2} + 2F_n^{\ 2} + F^2{}_{n-1}$$

Now by addition formula for Lucas numbers and Fibonacci numbers given above, we get

$$\frac{9_n^{\ 2} + L^2{}_n + 4F^2{}_{n-1}}{2} = L_{2n} + 2F^2{}_{n-1} + 2F_n^{\ 2} = L_{2n} + 2F_{2n-1}.$$

Now using

$M_n = L_n + 2F_{n-1}$, we obtain that

$$\frac{9_n^{\ 2} + L^2{}_n + 4F^2 n - 1}{2} = M_{2n}$$

Thus theorem follows by Theorem 6.

<u>Some Open Questions</u>.

(1) Are there any more triangular numbers in Mulatu numbers other than 1, 6, 28, and 45? If so, are they finite or infinite?

(2) Are there anymore Fermat numbers in Mulatu numbers other than 5 and 17? If so, are they finite or infinite?

(3) Are there any more Fibonacci numbers in Mulatu numbers other than 1 and 5? If so, are they finite or infinite?

(4) Are there any more Lucas numbers in Mulatu numbers other than 1 and 11? If so, are they finite or infinite?

(5) Observe that for n= 1,6,11, 16, and 21 all M, F, and L numbers have the same last digit. Is this pattern finite or infinite?

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