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TRIANGULAR NUMBERS IN ACTIONS

Aaliyah Walters and Mulatu Lemma

College of Science and Technology, Department of Mathematics, Savannah State University, USA.

Corresponding author: *Mulatu Lemma

Tel.:+1 ***Email:*** lemmam@savannahstate.edu

ABSTRACT

The triangular numbers are formed by partial sum of the series 1+2+3+4+5+6+7….+n In other words, triangular numbers are those counting numbers that can be written as = 1+2+3+…+ n. So,

T1= 1

T2= 1+2=3

T3= 1+2+3=6

T4= 1+2+3+4=10

T5= 1+2+3+4+5=15

T6= 1+2+3+4+5+6= 21

T7= 1+2+3+4+5+6+7= 28

T8= 1+2+3+4+5+6+7+8= 36

T9=1+2+3+4+5+6+7+8+9=45

T10 =1+2+3+4+5+6+7+8+9+10=55

In this paper we investigate some important properties of triangular numbers. Some important results dealing with the mathematical concept of triangular numbers will be proved. We try our best to give short and readable proofs. Most of the results are supplemented with examples.

KEYWORDS

Triangular numbers, Perfect square, Pascal Triangles, and perfect numbers.

INTRODUCTION AND BACKGROUND:

"Triangular numbers provide many wonderful contexts for mathematical thinking and problem solving. Triangular numbers are figurate numbers because they represent counting numbers as a geometric configuration of equally spaced points. This makes triangular number problems easy to present in a physically and visually engaging way that supports children to find generalizations and make connections with other figurate numbers such as square numbers and rectangular numbers (Samson, 2004)". Triangular numbers are a sequence that is arranged as an equilateral triangle. The numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45 and it continues on in that pattern. Triangular numbers are numbers that can make a "triangular dot pattern" which can be depicted as such:

In order to find a triangular number, you must add all consecutive numbers previous to the number shown. For example, if we use the number 4 and add its previous consecutive numbers AND itself (1+2+3+4) we get 10, thus making 10 a triangular; demonstrated in the above figure. These numbers helped to provide us with many fascinating discoveries that have shaped mathematics as we know it today.

Mathematicians have been fascinated for many years by the properties and patterns of triangular numbers [2]. We can easily hunt for triangular numbers using the formula:

The first 20 triangular numbers are as follows.

T1 1 T2 3 T3 6 T4 10 T5 15 T6 21 T7 28 T8 36 T9 45 T10 55 T11 66 T12 78 T13 91 T14 105 T15 120 T16 136 T17 153 T18 171 T19 190 T20 210

This can be shown in Figure 1 below.

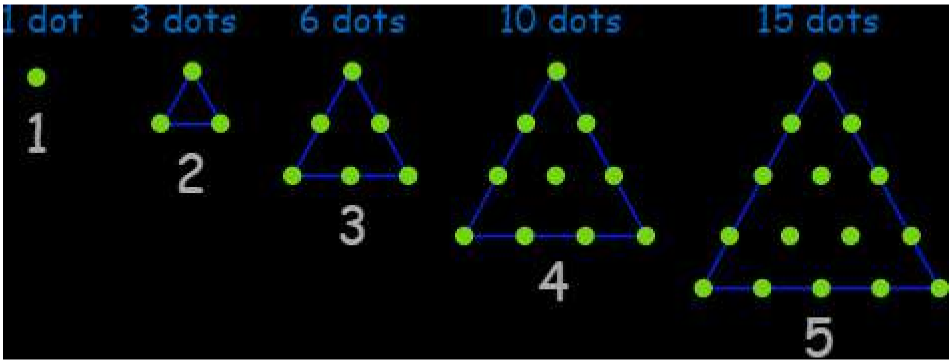


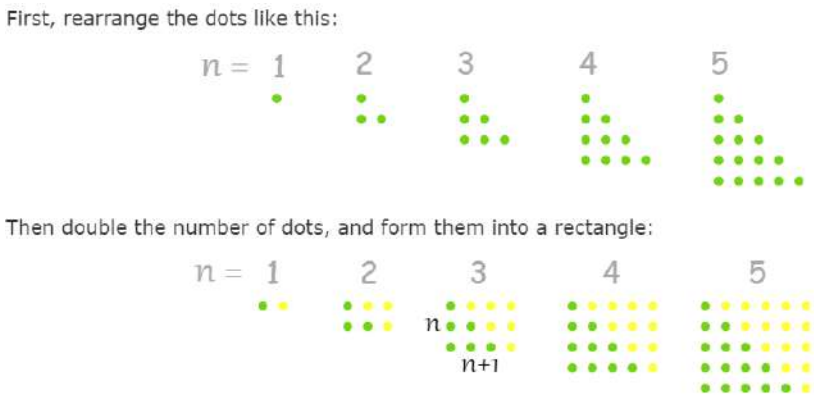
Figure 1. The Base of Triangular Numbers

To find out whether you have a triangular number, you pick a number and add all of the numbers that came before it. When doing this you must make sure to include the selected number with the previous ones. For example, you select the number 3, you would then add 1+2+3 which equal 6. This will make 6 a triangular number.

Due to this being a tedious process, in 1796 Gauss created a formula to help find out which larger numbers are triangular numbers.

T = (n)(n + 1) / 2

In which you would replace *n* with your selected number. This formula was created through putting the dots in triangles. Next, you would mirror the triangle and form a rectangle. Lastly, you would multiple the length x the width and divide by 2, because the triangle was doubled. This can be shown in Figure 2 below.



Main Findings

Theorem 1: T is a triangular number when 8T + 1 is a perfect square.

Proof. (I) ($\Rightarrow$) Assume T is a triangular number.

Let T = $\frac{n(n+1)}{2}$, with n as a positive integer.

$\Rightarrow$ 8T = $\frac{8n(n+1)}{2}$

$\Rightarrow$8T +1 = $\frac{8n(n+1)}{2}$ + 1

$\Rightarrow$8T +1 = $\frac{8n(n+1)}{2} + \frac{2}{2}$

$\Rightarrow$8T + 1 = $\frac{8n^2+8n+2}{2}$

$\Rightarrow$8T +1 = 2 $\frac{(4n^2+4n+1)}{2}$

$\Rightarrow$8T+1 = $(2n+1)(2n+1)$

$\Rightarrow$8T + 1 = $(2n+1)^2$

Hence, 8T is a perfect square.

(II) Assume 8T+1 is a perfect square. Then 8T +1 is odd $\Rightarrow$ for some n > 0, we have $8T+1=(2n+1)^2=4n^2+4n+2$ which implies that $T=\frac{n(n+1)}{2}$.

Hence T is a triangular number.

By (I) and (II) the theorem is proven.

Example 1. 6 is a triangular number

$$T = 6$$

$8(6) + 1 = 49$

$49 = 7 * 7$

Example 2. 11* 11 = 121

8T + 1 = perfect square

8(15) + 1 = 121

T = 15

15 is a triangular number.

Corollary 1. T is a triangular number when $n = \frac{\sqrt{8T+1}-1}{2}$ is an integer.

Proof: The corollary follows theorem 1 easily.

Theorem 2: If T_m and T_n are triangular numbers, then

$$T_{m+n} = T_m + T_n + mn$$

for m and n positive integers.

Proof:

Note: $T_m = \frac{m(m+1)}{2}$ & $T_n = \frac{n(n+1)}{2}$. Then

$$T_m + T_n + mn = \frac{m(m+1)}{2} + \frac{n(n+1)}{2} + mn$$

$$= \frac{m^2 + m + n^2 + n}{2} + mn$$

$$= \frac{m^2 + m + n^2 + n + 2mn}{2} = \frac{m^2 + 2mn + n^2 + m + n}{2}$$

$$= \frac{(m+n)(m+n) + (m+n)}{2} = \frac{(m+n)[m+n+1]}{2} = T_{m+n}$$

Example 3. Consider T_3 and T_4. Note that $T_3 = 6$ and $T_4 = 10$. Observe that $T_{3+4} = T_7 = 28$ and $T_3 + T_4 + 3(4) = 6 + 10 + 12 = 28$.

Hence, $T_{3+4} = T_3 + T_4 + 3(4)$

Theorem 3: If T_m and T_n are triangular numbers, then

$$T_{mn} = T_m T_n + T_{m-1} T_{n-1}$$

Proof: Note: $T_m = \frac{m(m+1)}{2}$ and $T_n = \frac{n(n+1)}{2}$. Then

$$T_m T_n + T_{m-1} T_{n-1} = \frac{m(m+1)}{2} \frac{n(n+1)}{2} + \frac{(m-1)m}{2} \frac{(n-1)n}{2}$$

$$= \left(\frac{m^2+m}{2}\right)\left(\frac{n^2+n}{2}\right) + \left(\frac{m^2-m}{2}\right)\left(\frac{n^2-n}{2}\right)$$

$$= \left[\frac{m^2n^2 + mn^2 + nm^2 + mn}{4}\right] + \left[\frac{m^2n^2 - mn^2 - nm^2 + mn}{4}\right]$$

$$= \frac{2m^2n^2 + 2mn}{4} = \frac{2mn(mn+1)}{4} = \frac{mn(mn+1)}{2}$$

$$= T_{mn}$$

Example 4. Let m =6 and n =7. Then $T_6 = 21$ and $T_7 = 28$.

By using $T_n = \frac{n(n+1)}{2}$, we get $T_{(6)(7)} = T_{42} = \frac{42(43)}{2} = 903$

We also have $T_5 T_6 = 15(21) = 315$ and $T_6 T_7 + T_5 T_6 =$ 588+315=903.

Hence, $T_{(6)(7)} = T_{42} = T_6 T_7 + T_5 + T_6$

Lemma 1. The sum of two consecutive triangular numbers is a perfect square

Proof: Let T_{n-1} and T_n be any two consecutive triangular numbers, such that

$$T_{n-1} = \frac{(n-1)(n)}{2} \text{ and } T_n = \frac{n(n+1)}{2}$$

Then,

$$T_{n-1} + T_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$$

$$= \frac{n^2 - n + n^2 + n}{2} = \frac{2n^2}{2} = n^2$$

which is a perfect square.

Example 5. Let T_6 and T_7 be any consecutive triangular numbers. Then $T_6 + T_7 = 21+28 = 49$, which is a perfect square.

Lemma 2. $1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Proof. We can easily prove the lemma using induction.

Example 6. Let k = 5 .Then $1^2 + 2^2 + 3^3 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$.

Also, we have $\frac{5(6)(11)}{6} = 55$ and hence $1^2 + 2^2 + 3^3 + 4^2 + 5^2 = \frac{5(6)(11)}{6}$.

Theorem 4. If T_k be triangular numbers for $k > 0$, then we have

$$\sum_{k=1}^{n} T_k = \frac{n(n+1)(n+2)}{6}$$

Proof: To prove the theorem, we apply *divide and conquer method* by considering two cases:

(1) If n is even, say n =2k, then

$$T_1 + T_2 + \ldots + T_n = (T_1 + T_2) + (T_3 + T_4) + \ldots + (T_{2k-1} + T_{2k})$$

$= 2^2 + 4^2 + ... + (2k)^2$ (by Lemma 1)

$= 4(1^2 + 2^2 + ... + k^2)$

$= \dfrac{4k(2k+1)(k+1)}{6}$ (by Lemma 2).

$= \dfrac{n(n+1)(n+2)}{6}$ as n = 2k

(2) If n is odd, say n= 2k+1, then

$$T_1 + T_2 + ... + T_n = (T_1 + T_2) + (T_3 + T_4) + ... + (T_{2k-1} + T_{2k}) + T_{2k+1}$$

$= 2^2 + 4^2 + ... + (2k)^2 + \dfrac{(2k+1)(2k+2)}{2}$ (**by Lemma 1 and definition of** T_k)

$= 4(1^2 + 2^2 + ... + k^2) + \dfrac{(2k+1)(2k+2)}{2}$

$= \dfrac{4k(2k+1)(k+1)}{6} + \dfrac{3(2k+1)(2k+2)}{6}$ (by Lemma 2.)

$= \dfrac{2k(2k+1)(2k+2)}{6} + \dfrac{3(2k+1)(2k+2)}{6}$

$= \dfrac{(2k+1)(2k+2)(2k+3)}{6}$

$= \dfrac{n(n+1)(n+2)}{6}$ as n = 2k+1

By (1) and (2) the Theorem is proved.

Example 7. Let n = 5. Then $\sum_{k=1}^{5} T_k = T_1 + T_2 + T_3 + T_4 + T_5$ = 1+3+6+10+15= 35. We also have,

$$\frac{5(6)(7)}{6} = 35 \text{ and hence } \sum_{k=1}^{5} T_k = \frac{5(6)(7)}{6}.$$

Theorem 5 For any natural number n**,** the number

$1 + 9 + 9^2 + 9^3 + \ldots + 9^n$ is a triangular number.

Proof: Let T =**$1 + 9 + 9^2 + 9^3 + \ldots + 9^n$**.Then **T=** $\frac{9^{n+1}-1}{8}$**.**

By **Theorem 1,** it is suffice to prove that **8T+1** is a perfect square. We will apply the divide and conquer method as in **Theorem 4.**

(1) If n is even, say n =2k, then

$$8T+1 = 8\left(\frac{9^{n+1}-1}{8}\right)+1 = 9^{n+1} = 9^{2k+1} = 9\left(9^{2k}\right) = \left(3^{2k+1}\right)^2,$$ which a perfect square.

(2) If n is odd, say n= 2k+1, then

$$8T+1 = 8\left(\frac{9^{n+1}-1}{8}\right)+1 = 9^{n+1} = 9^{2k+2} = \left(9^{k+1}\right)^2,$$ which a perfect square.

By (1)and (2), the theorem is proved.

Theorem 6. If T_n be triangular numbers for $n \geq 1$, then we have

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = 2$$

Proof: $\sum_{n=1}^{\infty} \frac{1}{T_n}$

$$=\sum_{n=1}^{\infty}\frac{2}{n(n+1)}$$

$$=2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 2(1) = 2$$

Proposition 2. The difference of the squares of two consecutive triangular numbers is a cube.

<u>Proof</u>: Consider $T_{n-1}=\frac{(n-1)n}{2}$ and $T_n=\frac{(n+1)n}{2}$

Then, $(T_n)^2-(T_{n-1})^2=\left(\frac{n(n+1}{2}\right)^2-\left(\frac{(n-1)n}{2}\right)^2$

$$=\frac{n^4+2n^2+n^2}{4}-\frac{n^4-2n^3+n^2}{4}=\frac{4n^3}{4}=n^3$$

Example 8. Let T_6 and T_7 be any two consecutive triangular numbers. Then

$(T_7)^2-(T_6)^2=28^2-21^2$ =(28+-21)(28-21)=(49)(7) = 7^3, which is a perfect cube.

Proposition 3: T is a triangular is number $\Rightarrow$ 9T+1 is a triangular number .

<u>Proof:</u> Assume T is a triangular number.

Let $T=\frac{n(n+1)}{2}$

$$\Rightarrow 9T=\frac{9n(n+1)}{2}$$

$$\Rightarrow 9T+1=\frac{9n(n+1)}{2}+1$$

$$=\frac{9n(n+1)}{2}+\frac{2}{2}$$

$$=\frac{9n(n+1)+2}{2}$$

$$= \frac{9n^2 + 9n + 2}{2}$$

$$= \frac{(3n+2)(3n+1)}{2}$$

$$= \frac{m(m+1)}{2}, \quad \text{where } m = \mathbf{3n+1}.$$

Hence, 9T+1 is a triangular number.

Example 9. Let T_8 be the triangular number. Then 9 T_8 +1= 45, which is a triangular number.

Proposition 4: n = $2^{k-1} + 2^k + 2^{k+1} + ... + 2^{2k-2}$ is a triangular number.

<u>Proof:</u> Note that n = $2^{k-1} + 2^k + ... + 2^{2k-2} = 2^{k-1}(1 + 2 + 2^2 + ... + 2^{k-1})$

$$= 2^{k-1}(2^k - 1)$$

$$= \frac{2^k(2^k - 1)}{2}$$

$$= \frac{m(m+1)}{2}, \quad \text{where } m = 2^k - 1.$$

Hence, n is a triangular number

Example 10. Let k =3. Then **n** = $2^{3-1} + 2^3 + 2^4$ = 28, which is a triangular number.

Proposition 5. $n = 1+2+3+4+...+(2^k - 1)$ is a triangular number**.**

<u>Proof</u> : Note that n =

$$\frac{2^k(2^k - 1)}{2}$$

$= \frac{m(m+1)}{2}$, where m = $2^k - 1$.

Hence, n is a triangular number

Example 11. Let k = 3. Then 1+2+3+4+5+6+7=28, which is a triangular numbers.

Proposition 6. Every Perfect number [3] is a triangular number.

Proof: Let n be a perfect number. Then n= $2^{k-1}(2^k - 1)$ where $2^k - 1$ is prime [3]. Note that n = $2^{k-1}(2^k - 1) = \frac{2^k(2^k - 1)}{2} = \frac{m(m+1)}{2}$, where m = $2^k - 1$. Hence n is a triangular number.

Proposition 7. Let T_n be a triangular number. Then:

(1) $\mathbf{T_n^2 = T_n + T_{n-1} * T_{n+1}}$

(2) $\mathbf{T_{n^2-1} = 2*T_n * T_{n-1}}$

Proof:

(1) We have $\mathbf{T_n + T_{n-1} * T_{n+1}} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} * \frac{(n+1)(n+2)}{2}$

$$= \frac{n(n+1)}{2} + \frac{n^4 + 2n^3 - n^2 - 2n}{4} \quad \frac{2n^2 + 2n + n^4 + 2n^3 - n^2 - 2n}{4}$$

$$= \frac{n^4 + 2n^3 + n^2}{4}$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$

$$= T_n^{\ 2}$$

(2) Note that $T_{n^2-1} = \frac{(n^2-1)(n^2)}{2}$

$$= \frac{((n-1)n)(n(n+1))}{2}$$

$$= \frac{2((n-1)n)(n(n+1))}{4}$$

$$= 2*\frac{(n-1)n}{2}*\frac{n(n+1)}{2} = 2*T_{n-1}*T_n$$

<u>Open Questions</u>:

The following are lists of open questions that we found.

1. Other than 120 is there another triangular number which can be expressed as the product of five consecutive numbers?

Note: 120= 1*2*3*4*5

2. Other 9 (refer to Theorem 5) is there any other positive integer n such that $1+n+n^2+n^3+\ldots+n^k$ is a triangular number?

3. Other than 6 is there any other positive integer N such that N, 11N, 111N are all triangular numbers? Note that 6, 66, 666 are triangular numbers.

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