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Our Interesting Journey with the Fascinating Mathematics of the Closed-form Formula of Riemann Zeta and Eta Functions

Mulatu Lemma, Agegnehu Atenaand Tilahun Muche & Wondimu Tekalign

College of Science and Technology, Savannah State University, Savannah, GA 3104, USA.

Corresponding author: *Mulatu Lemma
Tel.:+1 Email: lemmam@savannahstate.edu

ABSTRACT

An explicit identity of sums of powers of complex functions presented via this a closed-form formula of Riemann zeta function produced at any given non-zero complex numbers. The closed-form formula showed us Riemann zeta function has no unique solution for any given non-zero complex numbers which means Riemann zeta function is entirely divergent. Infinitely many zeros of Riemann zeta function produced unfortunately those zeros also gives us non-zero values of Riemann zeta function. Among those zeros, some of them are zeros of Riemann hypothesis. The present paper also discussed on eta function (alternating Riemann zeta function) with exactly the same behavior as Riemann zeta function.

KEYWORDS

Sums of Powers of Complex Functions, Riemann zeta and Eta functions.

Introduction

In his 1859 paper on the number of primes less than a given magnitude, bernhard Riemann

(1826-1866) examined the properties of the function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

For a complex number. This function is analytic for real part of **s** greater than 1 and is related to the prime numbers by the euler product formula

$$\zeta(s) = \prod_{p(prime)} (1 - p^{-s})^{-1}$$

again defined for real part of **s** greater than one. This function extends to points with real part **s** greater than zero by the formula (among others)

$$\zeta(s) = (1 - 2^{(1-s)})^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$$

The Riemann Hypothesis

The zeta function has no zeros in the region where the real part of **s** is greater than or equal to one. In the region with real part of **s** less than or equal to zero the zeta function has zeros at the negative even integers; these are known as the trivial zeros. All remaining zeros lie in the strip where the real part of **s** is strictly between **0** and **1** (**the critical strip**). It is known that there are infinitely many zeros on the line **1/2+it** as **t** ranges over the real numbers. This line in the complex plane is known as the **critical line**. The Riemann Hypothesis (**RH**) is that all non-trivial zeros of the zeta function lie on the critical line [1],[6],[7]. Let's say that again:

Riemann Hypothesis:

All non-trivial zeros of the zeta functions lie on the line **1/2+it** as **t** ranges over the real numbers.

The Functional Equation

The functional equation of the zeta function is

$$\zeta(s) = \Gamma(1-s)(2\pi)^{s-1} 2 \sin\left(\frac{\pi s}{2}\right) \zeta(1-s)$$

From which values of the zeta function at **s** can be computed from its values at **(1-s)**. Using this equation one sees immediately that the zeta function is zero at the negative even integers [3],[5],[11],[12].

Many famous mathematicians studied and developed equations to prove Riemann Hypothesis in different approach [2],[8],[9]. An analytic continuation of Riemann zeta function developed which agrees for all complex numbers [4].

The present paper gives us closed-form formula of Riemann zeta function which spits out non-unique solutions for each complex numbers excepting zero.

2. Four Definitions

Throughout this paper we will use the following definitions.

Definition 1. *For every* $k \in \mathbb{Z}^+ | \{1\}$, $a_1, a_2 \in \mathbb{R}$ *and* $d_1, d_2 \in \mathbb{R} | \{0\}$, *define the following*

$$B^1 := B^1(k, a_1, a_2, d_1, d_2) = \frac{1}{\ln(k)} \arctan\left(\frac{a_2 + (k-1)d_2}{a_1 + (k-1)d_1}\right) \tag{1}$$

$$A^1 := A^1(k, a_1, d_1, B^1) = \frac{1}{\ln(k)} \ln\left(\frac{a_1 + (k-1)d_1}{\cos(B^1 \ln(k))}\right) \tag{2}$$

$$A^2 := A^2(k, a_2, d_2, B^1) = \frac{1}{\ln(k)} \ln\left(\frac{a_2 + (k-1)d_2}{\sin(B^1 \ln(k))}\right) \tag{3}$$

Definition 2. *Define the Riemann zeta function,* $\zeta(s)$, *for all complex numbers, s with real part greater than one by:*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Definition 3. *Define the eta function(alternating Riemann zeta function),* $\eta(s)$, *for all complex numbers, s with real part greater than zero by:*

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$$

Definition 4. *Let* **s** *be a complex number with real part of* **s** *greater than zero. Then we define the gamma function,* $\Gamma(s)$ *as*

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$$

Note that $\Gamma(s+1) = s\Gamma(s)$ *with real part of a complex number* **s** *greater than zero.*

3. Two Theorems

Theorem 1. *[10] For every complex numbers a and d, where* $d = 06$ *, then we have*

$$\frac{d}{(n-1)!(n-3)!} \sum_{r=1}^{k} (a + (r-1)d)^{n-1} = \sum_{i=0}^{n-3} \frac{1}{i!(n-i)!(n-3-i)!} \left(\frac{d}{2}\right)^i (-1)^i S_{n-i} \tag{4}$$

Where

$$S_{n-i} = \left(\frac{n-i}{2} - 1\right) k d^{n-i} - \frac{n-i}{2} d^{n-i-2}((a+kd)^2 - a^2) + (a+kd)^{n-i} - a^{n-i} \tag{5}$$

Theorem 2. *[10]For every complex numbers a and d, where d* = 06 *, then we have*

$$\sum_{r=1}^{k}(-1)^{(r-1)}(a+(r-1)d)^{n-1} = \frac{1}{nd}\left[\sum_{i=0}^{n-3}\binom{n-3}{i}\left(\frac{d}{2}\right)^{i}\frac{n!}{(n-i)!}(-1)^{(i+1)}L_{n-i}\right] \qquad (6)$$

Where

$$L_{n-i} = \left(\frac{n-i}{2}-1\right)kd^{n-i}+\frac{n-i}{2}d^{n-i-2}((a+kd-d)^2-(a-d)^2)+(-1)^{(n-i-1)}[(a+kd-d)^{n-i}-(a-d)^{n-i}] \qquad (7)$$

4. Main Result

Theorem 3. *Define a complex numbers* $a = a_1 + a_2 i$, $d = d_1 + d_2 i$ 6= (0 ,0), $s = A^1 + iB^1$ *or* $s = A^2 + iB^1$ *and* 1 =6 $x \in \mathbb{R}^+$. *If*

$$a + (x-1)d = x^s$$

$$B^1 := \frac{1}{\ln(x)}\arctan\left(\frac{a_2+(x-1)d_2}{a_1+(x-1)d_1}\right)$$

$$A^1 := \frac{1}{\ln(x)}\ln\left(\frac{a_1+(x-1)d_1}{\cos(B^1\ln(x))}\right)$$

$$A^2 := \frac{1}{\ln(x)}\ln\left(\frac{a_2+(x-1)d_2}{\sin(B^1\ln(x))}\right)$$

Then

Proof

$$x^s = a + (x-1)d$$

$$\Rightarrow e^{\ln(x^s)} = a + (x-1)d$$

$$\Rightarrow e^{s\ln(x)} = a + (x-1)d$$

$$\Rightarrow e^{(A+Bi)\ln(x)} = a + (x-1)d$$

$$\Rightarrow e^{(A\ln(x)+iB\ln(x))} = a + (x-1)d$$

$$\Rightarrow e^{A\ln(x)}e^{iB\ln(x)} = a + (x-1)d$$

$$\Rightarrow e^{A\ln(x)}\Big(\cos(B\ln(x)) + i\sin(B\ln(x))\Big) = a + (x-1)d$$

$$\Rightarrow e^{A\ln(x)}\cos(B\ln(x)) + ie^{A\ln(x)}\sin(B\ln(x)) = a + (x-1)d$$

$$\Rightarrow e^{A\ln(x)}\cos(B\ln(x)) + ie^{A\ln(x)}\sin(B\ln(x)) = a_1 + a_2 i + (x-1)(d_1 + d_2 i)$$

$$\Rightarrow e^{A\ln(x)}\cos(B\ln(x)) + ie^{A\ln(x)}\sin(B\ln(x)) = a_1 + (x-1)d_1 + (a_2 + (x-1)d_2)i$$

$$\Rightarrow e^{A\ln(x)}\cos(B\ln(x)) = a_1 + (x-1)d_1 \quad (8)$$

$$\Rightarrow e^{A\ln(x)}\sin(B\ln(x)) = a_2 + (x-1)d_2 \quad (9)$$

Now divide equation (9) by equation (8), then we have

$$\tan(B\ln(x)) = \frac{\sin(B\ln(x))}{\cos(B\ln(x))} = \frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1}$$

$$\Rightarrow B = \frac{1}{\ln(x)}\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1}\right)$$

We can let that

$$B^1 = B^1(x, a_1, a_2, d_1, d_2) = B = \frac{1}{\ln(x)}\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1}\right)$$

$$\Rightarrow B^1 = B^1(x, a_1, a_2, d_1, d_2) = \frac{1}{\ln(x)}\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1}\right)$$

From equation (8), we have

$$e^{A\ln(x)}\cos(B^1\ln(x)) = a_1 + (x-1)d_1$$

$$\Rightarrow e^{A\ln(x)} = \frac{a_1 + (x-1)d_1}{\cos(B^1\ln(x))}$$

$$\Rightarrow A\ln(x) = \ln\left(e^{A\ln(x)}\right) = \ln\left(\frac{a_1 + (x-1)d_1}{\cos(B^1\ln(x))}\right)$$

$$\Rightarrow A = \frac{1}{\ln(x)}\ln\left(\frac{a_1 + (x-1)d_1}{\cos(B^1\ln(x))}\right)$$

We can let that

$$A^1 := A^1(x, a_1, d_1, B^1) = A = \frac{1}{\ln(x)}\ln\left(\frac{a_1 + (x-1)d_1}{\cos(B^1\ln(x))}\right)$$

$$\Rightarrow A^1 := A^1(x, a_1, d_1, B^1) = \frac{1}{\ln(x)}\ln\left(\frac{a_1 + (x-1)d_1}{\cos(B^1\ln(x))}\right)$$

And now from equation (9), we have

$$e^{A\ln(x)}\sin(B^1\ln(x)) = a_2 + (x-1)d_2$$

$$\Rightarrow A = \frac{1}{\ln(x)}\ln\left(\frac{a_2 + (x-1)d_2}{\sin(B^1\ln(x))}\right)$$

We can let that

$$A^2 := A^2(x, a_2, d_2, B^1) = A = \frac{1}{\ln(x)}\ln\left(\frac{a_2 + (x-1)d_2}{\sin(B^1\ln(x))}\right)$$

$$\Rightarrow A^2 := A^2(x, a_2, d_2, B^1) = \frac{1}{\ln(x)}\ln\left(\frac{a_2 + (x-1)d_2}{\sin(B^1\ln(x))}\right)$$

Theorem 4. *If* $s = A^1 + B^1 i$ *or* $s = A^2 + B^1 i$*, then for every* 1 =6 $x \in \mathrm{R}^+$

$$x^s$$

$$= \frac{a_1 + (x-1)d_1}{\cos(B^1 \ln(x))} \Bigg[\cos\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) + i \sin\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) \Bigg]$$

or

$$x^s$$

$$= \frac{a_2 + (x-1)d_2}{\sin(B^1 \ln(x))} \Bigg[\cos\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) + i \sin\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) \Bigg]$$

Proof

$$x^s = e^{\ln(x^s)} = e^{s\ln(x)} = e^{(A^1 + B^1 i)\ln(x)} = e^{A^1 \ln(x)} e^{iB^1 \ln(x)}$$

$$\Rightarrow x^s = e^{A^1 \ln(x)} e^{iB^1 \ln(x)} = e^{\left(\ln\left(\frac{a_1 + (x-1)d_1}{\cos(B^1 \ln(x))} \right) \right)} e^{iB^1 \ln(x)}$$

$$\Rightarrow x^s = \frac{a_1 + (x-1)d_1}{\cos(B^1 \ln(x))} e^{iB^1 \ln(x)}$$

$$\Rightarrow x^s = \frac{a_1 + (x-1)d_1}{\cos(B^1 \ln(x))} \Big(\cos(B^1 \ln(x)) + i \sin(B^1 \ln(x)) \Big)$$

Hence

$$x^s$$

$$= \frac{a_1 + (x-1)d_1}{\cos(B^1 \ln(x))} \Bigg[\cos\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) + i \sin\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) \Bigg]$$

and

$$x^s = e^{A^2 \ln(x)} e^{iB^1 \ln(x)} = e^{\left(\ln\left(\frac{a_2 + (x-1)d_2}{\sin(B^1 \ln(x))} \right) \right)} e^{iB^1 \ln(x)} \qquad (10)$$

Hence,

$$x^s$$

$$= \frac{a_2 + (x-1)d_2}{\sin(B^1 \ln(x))} \left[\cos\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) + i \sin\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) \right] \tag{11}$$

Theorem 5. *If* $s = A^1 + B^1 i$ *or* $s = A^2 + B^1 i$, *then for every* $x \in \mathrm{R}^+$

$$x^{-s}$$

$$= \frac{\cos(B^1 \ln(x))}{a_1 + (x-1)d_1} \left[\cos\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) - i \sin\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) \right]$$

$$x^{-s}$$

$$= -\frac{\sin(B^1 \ln(x))}{a \quad x \quad d} \left[\cos\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a \quad x \quad d} \right) \right) \right.$$

or

$${}_2 + (\quad -1) \;{}_2 \qquad\qquad {}_1 + (\quad -1) \;{}_1$$

$$\left. - i \sin\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1} \right) \right) \right]$$

$$x^{-s} = e^{\ln(x^{-s})} = e^{-s \ln(x)} = e^{(-A^1 - B^1 i)\ln(x)} = e^{-A^1 \ln(x)} e^{-iB^1 \ln(x)}$$

$$\Rightarrow x^{-s} = e^{-A^1 \ln(x)} e^{-iB^1 \ln(x)} = e^{\left(-\ln\left(\frac{a_1 + (x-1)d_1}{\cos(B^1 \ln(x))} \right) \right)} e^{-iB^1 \ln(x)}$$

proof

$$\Rightarrow x^{-s} = \left[\frac{a_1 + (x-1)d_1}{\cos(B^1 \ln(x))}\right]^{-1} e^{-iB^1 \ln(x)}$$

$$\Rightarrow x^{-s} = \frac{\cos(B^1 \ln(x))}{a_1 + (x-1)d_1}\Big(\cos(B^1 \ln(x)) - i\sin(B^1 \ln(x))\Big)$$

$$x^{-s}$$

$$= \frac{\cos(B^1 \ln(x))}{d}\left[\cos\left(\arctan\left(\frac{a_2 + (x-1)d_2}{d}\right)\right)\right.$$

Therefore

$$a_1 + (x-1) \quad {}_1 \qquad\qquad a_1 + (x-1) \quad {}_1$$

$$\left.- i\sin\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1}\right)\right)\right] \tag{12}$$

And

$$x^{-s} = e^{-A^2 \ln(x)} e^{-iB^1 \ln(x)} = e^{\left(-\ln\left(\frac{a_2 + (x-1)d_2}{\sin(B^1 \ln(x))}\right)\right)} e^{-iB^1 \ln(x)}$$

Hence

$$x^{-s}$$

$$= -\frac{\sin(B^1 \ln(x))}{a_2 + (x-1)d_2}\left[\cos\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1}\right)\right)\right.$$

$$\left.- i\sin\left(\arctan\left(\frac{a_2 + (x-1)d_2}{a_1 + (x-1)d_1}\right)\right)\right] \tag{13}$$

References:

J. Brian Conrey, The Riemann Hypothesis, Notices of the AMS, Volume 50, Number 3, March 2003.

Iaroslav V. Blagouchine, Three Notes on Ser's and Hasse's Representations for Zeta Functions, INTEGERS 18A (2018).

Stephen S. Gelbart and Stephen D. Miller, Riemann's Zeta Function and Beyond, Bulletin (New Series) of the American Mathematical Society, Volume 41, Number 1, Pages 59-112, October 30, 2003.

J. Sondow, Analytic continuation of Riemanns zeta function and values at negative integers via Eulers transformation of series, Proc. Amer. Math. Soc., 120 (1994),421424.

M. O. Rubinstein, Identities for the Riemann zeta function, Ramanujan J., 27 (2012), 2942.

L. Fekih-Ahmed, On the zeros of the Riemann zeta function, arXiv:1004.4143 (2010).

D. F. Connon, Some possible approaches to the Riemann hypothesis via the Li/Keiper constants, arXiv:1002.3484 (2010).

D. F. Connon, Some series and integrals involving the Riemann zeta function, binomial coefficients and the harmonic numbers. Volume II(a), arXiv:0710.4023 (2007).

D. F. Connon, Some series and integrals involving the Riemann zeta function, binomial coefficients and the harmonic numbers. Volume II(b), arXiv:0710.4024 (2007).

D. J. Negash, An Explicit Identity to solve sums of Powers of Complex Functions. arXiv:1811.10860v3 [math.HO] (2018).

R. Askey, The q-gamma and q-beta functions, Applicable Analysis, vol. 8, no. 2,pp. 125141, 1978.

E. Freitag and R. Busam, Complex analysis, ser. Universitext. Springer-Verlag, Berlin, 2005, Translated from the 2005 German edition by Dan Fulea.