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Our Brief Journey with properties and patterns of Triangular Numbers

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Abstract

The triangular numbers are formed by partial sum of the series 1+2+3+4+5+6+7…. +n
In other words, triangular numbers are those counting numbers that can be written as = 1+2+3+…+ n. So,

T1= 1

T2= 1+2=3

T3= 1+2+3=6

T4= 1+2+3+4=10

T5= 1+2+3+4+5=15

T6= 1+2+3+4+5+6= 21

T7= 1+2+3+4+5+6+7= 28

T8= 1+2+3+4+5+6+7+8= 36

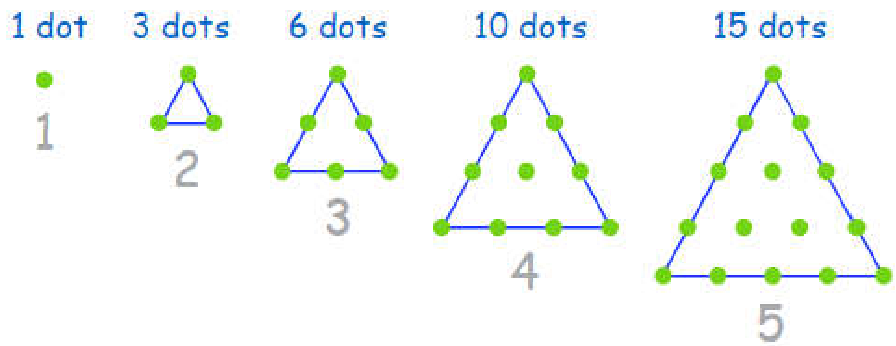
T9=1+2+3+4+5+6+7+8+9=45

T10 =1+2+3+4+5+6+7+8+9+10=55

In this paper some important properties of triangular numbers are studied.

Introduction and Background:

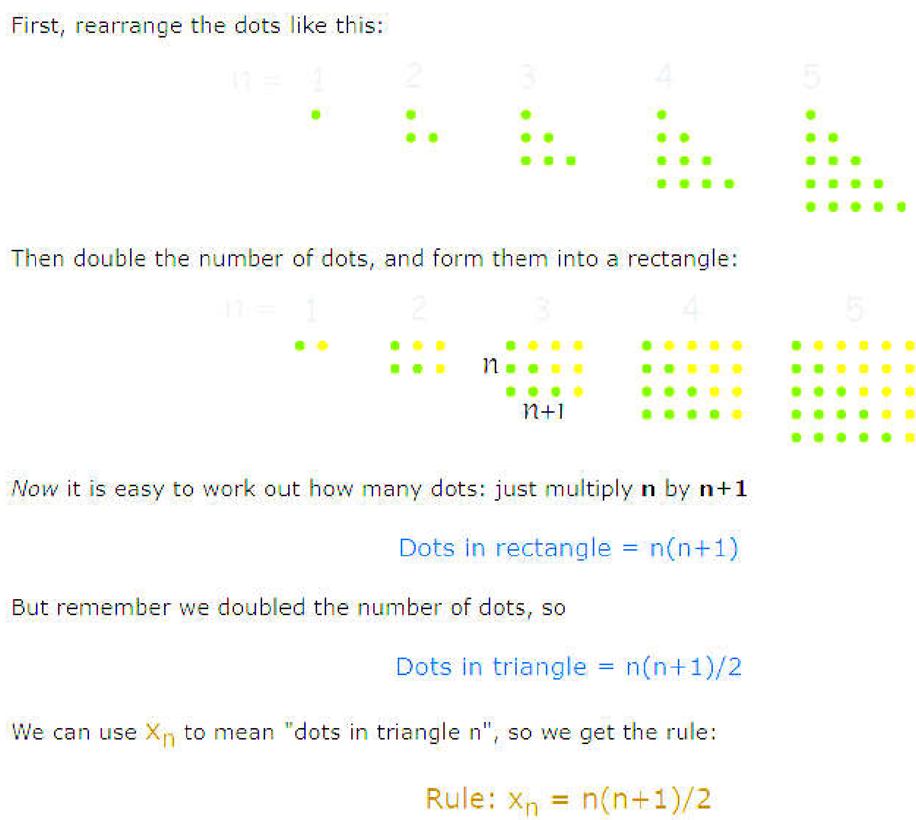
"Triangular numbers provide many wonderful contexts for mathematical thinking and problem solving. Triangular numbers are figurate numbers because they represent counting numbers as a geometric configuration of equally spaced points. This makes triangular number problems easy to present in a physically and visually engaging way that supports children to find generalizations and make connections with other figurate numbers such as square numbers and rectangular numbers (Samson, 2004)". Triangular numbers are a sequence that are arranged as an equilateral triangle. The numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45 and it continues on in that pattern. Triangular numbers are numbers that can make a "triangular dot pattern" which can be depicted as such:



In order to find a triangular number, you must add all consecutive numbers previous to the number shown. For example, if we use the number 4 and add its previous consecutive numbers AND itself (1+2+3+4) we get 10, thus making 10 a triangular; demonstrated in the above figure. These numbers helped to provide us with many fascinating discoveries that have shaped mathematics as we know it today.

The mathematics of Triangular Numbers

It is quite easy to create visual and pictorial diagram to represent these numbers and the rules associated with them. You must first arrange your dots into triangular form. Then you are to double the dots and form a rectangle. Once you do that, using algebraic methods you are able to form a rule which is demonstrated in the figure below:



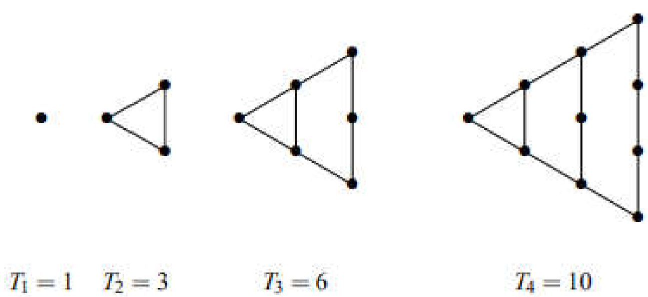
This rule/equation is what you can use to find any triangular number you desire to know and its place in the sequence. For example, if I wanted to know what the 87th Triangular Number was, I would simply replace “n” with “87” and complete your calculations. A common misconception that some may get Triangular Numbers with is “factorials”. The difference is with Triangular Numbers you add each of the numbers and in factorial, you multiply.

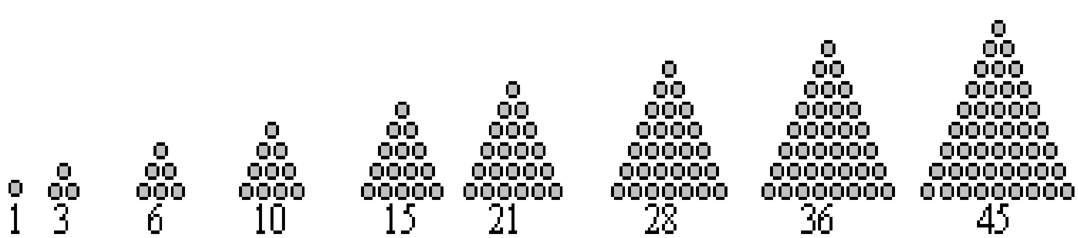
Triangular numbers were stumbled upon by a young boy named Carl Friedrich Gauss who was later given the name “Prince of Mathematics”. Carl F. Gauss was born on April 30th, 1777 in Germany and lived until 1855. Gauss was not only an expert mathematician but he was also known for contributions in Geodesy, electromagnetism and Planetary Astronomy. He was the only child from his parents but he had an extraordinary gift when it came to mathematics and retaining different elaborate calculations. His first significant mathematical discovery came when he was only fifteen years old. He discovered that a regular polygon with 17 sides can be constructed by only using a ruler and compass. This discovery along with its extensive and detailed proof helped future mathematicians such as Galois because of the scholarly analysis of the factorization of polynomial equations.

When he was a young boy, his teacher instructed the class to add up numbers from one to 100; a task that the teacher thought would take a long time for the young students to do. On the contrary, young Carl Gauss saw a pattern and shortly raised his hand and told his teacher that he was finished. Shocked, the teacher looked on Gauss paper and realized that he had in fact figured out a pattern in which to add consecutive numbers. The example he created in table format is:

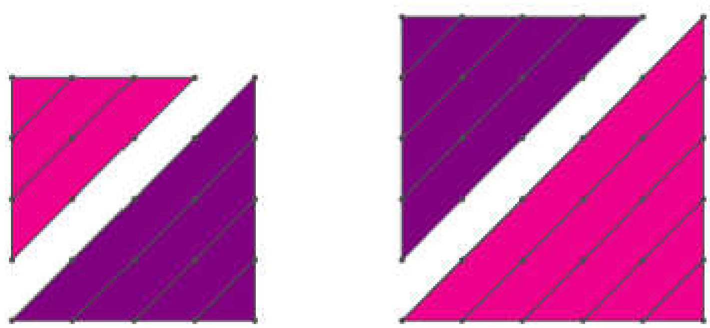
1	2	3	4	5	...	99	100
100	99	98	97	96	...	2	1
101	**101**	**101**	**101**	**101**	**...**	**101**	**101**

He realized that if you write numbers from 1-100, then put another row underneath with the numbers in descending order, each number summed to be 101. He then added 101 a hundred times in order to get 10,100 then divided by two (because he used the set of numbers twice) and came up with 5050. Well now that poses the question: how does this relate to Triangular Numbers? By finding what the sum of each number added together between 1 and 100, he found the 100th triangular number. From his hypothesis, we can come to the conclusion that$T(n) = \frac{n(n+1)}{2}$. With this formula, you can find any Triangular Number by simply plugging in the number you want to find. "Triangular numbers were known to the ancient Greeks and were viewed by them with mystical reverence. The triangular number 10 was considered to be a symbol of 'perfection', being the sum of 1 (a point), 2 (a line), 3 (a plane) and 4 (a solid)".Below you will see a two diagrams showing the first 9 Triangular Numbers:





Perfect squares are numbers that are closely related to Triangular Numbers and there are many mathematical properties that were found using Triangular Numbers and perfect Squares. There are numbers called “Square Triangular Numbers” which means that number is both a Triangular Number and a Perfect Square. These numbers include 0, 1, 36, 1225 and another infinite amount of numbers. Another interesting fact about Triangular Numbers is that if you calculate the sum of consecutive Triangular Numbers; you will get a Perfect Square. The Greeks are the ones who discovered this interesting fact and used an algebraic proof.There is a more obvious depiction of this proof that uses no words at all and it is below:



$5^2 = T_4 + T_5$ $\qquad$ $6^2 = T_5 + T_6$

For example, if you add the third Triangular Number which is “6” and the fourth Triangular Number which is “10”; you get 16. This is a simple concept yet it is very interesting that somebody figured this out. This concept helps with quick calculations for different triangular numbers. If I wanted to know what two triangular numbers combine to get 17^2, I would simply need to set up the equation $17^2 = T_{16} + T_{17}$.

Another interesting pattern when it comes to Triangular Numbers is that every two numbers, the numbers switch between even and odd as illustrated below:

1	3	6	10	15	21	28	36	45	55
66	78	91	105	120	136	153	171	190	210
231	253	276	300	325	351	378	406	435	465
496	528	561	595	630	666	703	741	780	820
861	903	946	990	1035	1081	1128	1176	1225	1275
1326	1378	1431	1485	1540	1596	1653	1711	1770	1830
1891	1953	2016	2080	2145	2211	2278	2346	2415	2485
2556	2628	2701	2775	2850	2926	3003	3081	3160	3240
3321	3403	3486	3570	3655	3741	3828	3916	4005	4095
4186	4278	4371	4465	4560	4656	4753	4851	4950	5050

Square numbers (numbers that are both square and triangular) can be written as n^2 for some n and the equation $\mathrm{n}^2 = \frac{\mathrm{k(k+1)}}{2}$ which is also known as a Diophantine Equation. This equation can be solved algebraically and manipulating variables. The proof is below (by clicking the picture, it will take you to the source):

A number which is simultaneously square and triangular. Let T_n denote the nth triangular number and S_m the mth square number, then a number which is both triangular and square satisfies the equation $T_n = S_m$, or

$$\frac{1}{2} n(n+1) = m^2. \tag{1}$$

Completing the square gives

$$\begin{aligned} \tfrac{1}{2}(n^2+n) &= \tfrac{1}{2}\left(n+\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) && (2) \\ &= m^2 && (3) \\ \tfrac{1}{8}(2n+1)^2 - \tfrac{1}{8} &= m^2 && (4) \\ (2n+1)^2 - 8m^2 &= 1. && (5) \end{aligned}$$

Therefore, defining

$$\begin{aligned} x &\equiv 2n+1 && (6) \\ y &\equiv 2m && (7) \end{aligned}$$

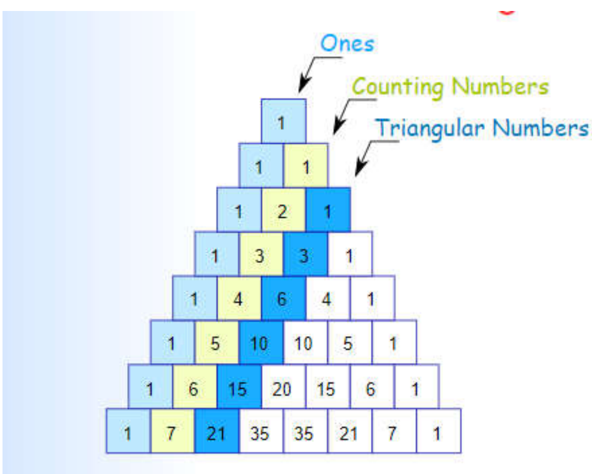
gives the Pell equation

$$x^2 - 2y^2 = 1 \tag{8}$$

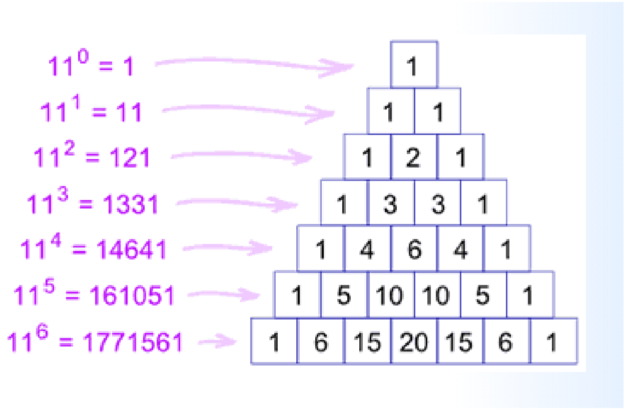
(Conway and Guy 1996). The first few solutions are $(x, y) = (3, 2)$, (17, 12), (99, 70), (577, 408), ... These give the solutions $(n, m) = (1, 1)$, (8, 6), (49, 35), (288, 204), ... (OEIS A001108 and A001109), corresponding to the triangular square numbers 1, 36, 1225, 41616, 1413721, 48024900, ... (OEIS A001110; Pietenpol 1962). In 1730, Euler showed that there are an infinite number of such solutions (Dickson 2005).

There are several other interesting patterns and cases that involve Triangular Numbers because they are very versatile numbers. With continued research, we will continue to discover more interesting facts about these Triangular Numbers.

Pascal's Triangle is another mathematical pattern that is closely related to Triangular Numbers. Pascal's Triangle contains the values of the binomial coefficient and is named after Blaise Pascal. In order to build the triangle, you must start with a 1 at the top the triangle and write one's around the edge of the triangle. Following that, you will notice that each number is calculated by adding the number above. The interesting parts of the Pascal Triangle is the unique properties that each line and diagonal possess. For example, the first diagonal is just one's (notice how we stated that the outside of the triangle is just one's). The second diagonal is counting numbers and the third diagonal is Triangular Numbers. This in itself is another easy way to calculate Triangular Numbers and it shows their versatility and impact in different mathematical patterns. The diagram below demonstrates the construction and diagonals of the Pascal's Triangle:



Other unique properties about Pascal's Triangle is the horizontal sums and how each row is the exponent of 11. If you add each row up individually you will notice that each sum is doubled every time you go to the next row. For example, the first row is 1, the sum of the second row is two, the sum of the third row is four and so on and so forth. As far as the exponent of 11; each row is the exponent of 11 best depicted below:



This brings us to Binomial Coefficients which are closely related to Triangular numbers. Binomial coefficients are defined as the positive integers that a relevant in the Binomial Theorem. The binomial coefficient is represented by $\binom{n}{k}$ and is read as "the number of ways of picking "k" unordered outcomes from "n" possibilities" (Weissman). This can also be represented by the Combination Formula where $\binom{n}{k}$ or ${}_nC_k$. The numbers calculated from this formula are referred to as Binomial Coefficients because they are also used in the Binomial Theorem (for any positive integer *n*, the *n*th power of the sum of two numbers '*a"* and "*b"* may be expressed as the sum of *n* + 1 terms of the form) (Gregerson). Using the Pascal's Triangle, we see that if we input the row number for "n" and element for "k", it will calculate to be in that position on the triangle as shown below as well as the formula for binomial coefficients:

$\binom{n}{r}$ refers to the nth row, rth element in Pascal's Triangle.

0th row						1						
1st row					1		1					
2nd row				1		2		1				
3rd row			1		3		3		1			
4th row		1		4		6		4		1		
5th row	1		5		10		10		5		1	
	0th		1st		2nd		3rd		4th		5th	

Elements of the 5th row of Pascal's Triangle

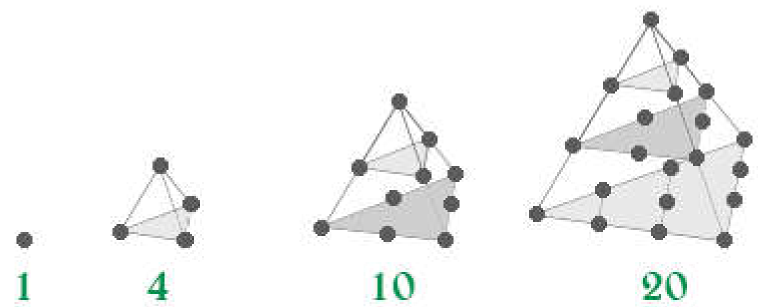
Example: $\binom{5}{4}$ is the 5th row, 4th element, so $\binom{5}{4} = 5$.

${}_5C_4$ is another notation for the same element.

For non-negative integer values of n (number in the set) and k (number of items you choose), every binomial coefficient ${}_nC_k$ is given by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

One interesting fact about Triangular numbers is their relationship to perfect numbers. A perfect number is a positive integer that is equal to the sum of its divisors such as 6, 28 and 496. What that means is, if you take the divisors of 28 (1, 2, 4, 7, 14) and add them together, you get 28. This is related to triangular numbers because all even perfect numbers are triangular. Interestingly enough, all perfect numbers are in fact even which then concludes that all perfect numbers are triangular numbers. We discussed earlier how adding consecutive triangular numbers would equate to a perfect square. Another fascinating fact about Triangular numbers is that if you add consecutive triangular numbers, it produces tetrahedral numbers. For example, the first 5 triangular numbers are 1, 3, 6, 10 and 15. If you add 1 and 3, you get 4 which is the second tetrahedral number. If you add 1, 3 and 6; then you get 10 and so on and so forth.



Where triangular numbers obviously are depicted as a triangle, tetrahedral numbers are depicted as a pyramid. This shows that from a mathematical level and pictorial level, that triangles and pyramids are in the same family.

Triangular numbers are so versatile in how they are used and he vast amount of mathematical fields that they are involved in. Algebraically, binomial coefficients are used for determining different permutations, combinations and probabilities. Geometrically there are many different ways to manipulate triangular and rectangular proofs by using theorems derived from triangular numbers. The simplicity of Triangular numbers is fascinating because of how its DNA is in all different aspects of mathematics. It's amazing that a young boy in the 1800s derived a simple pattern for what seemed to be, complex question that would impact the mathematical world so heavily.

Acknowledgements:

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