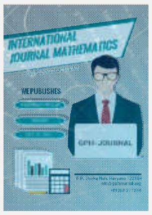

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Our Perfect Brief Journey with Perfect Numbers

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Abstract

Perfect numbers have been a fascination of mathematicians for centuries. This is due to their realization of the pattern and properties that create the numbers 6, 28, 496, and 8128. These numbers are known as perfect numbers for being the sum of their factors, also known as its proper positive divisors. In this paper we will dive into the history of perfect numbers, as well as its properties, theorems, and proofs for which all play an important role in the theory of even perfect numbers.

Keywords

Prime Numbers, Perfect Numbers, Triangular Numbers, Natural Numbers, Pascal's Triangle, Binomial Coefficient, Combination, Figurate Number.

Introduction and Background

There has been interest in perfect numbers since ancient times, possibly even since numbers first piqued curiosity. This is known because of the knowledge and documentation based around the Pythagoreans from ancient times who equated the perfect number 6 to marriage, beauty, and health. Others with an interest in perfect numbers include Nicomachus, Rabbi Josef Jehuda Ankin, Erycius Puteanus, Saint Augustine, Philo Judeus, Descartes, and most importantly Euclid and Euler. Euclid was a Greek mathematician who is known for categorizing perfect numbers, but years later Euler (a Swiss mathematician) created the converse to Euclid's Theorem.

Theorem I (Euclid): *If* ^{n} is *prime, then* $N = 2^{n-1}(2^n - 1)$ *is perfect.*

Proof: We will show that N = the sum of its proper factors.

We will find all of the proper factors of $2^{n-1}(2^n - 1)$, and add them together.

Since ^{n} Is prime, let it be set as p. Meaning, $p = 2^n -$. Then, $N = p(2^{n-1})$.

Let's list all factors of 2^{n-1}, as well as other proper factors of N.

Factorsof2^{n-1}	Other Proper Factors
1	
2	$2p$
2^2	2^2p
.	.
.	.
.	.
2^{n-1}	$2^{n-2}p$

Adding the first column gives the result:

$$1 + 2 + 2^2 + 2^3 \ldots + 2^{n-1} = 2^n - 1 = p$$

And when the same is done to the second column, the result is:

$$p + 2p + 2^2p + 2^3p \ldots + 2^{n-2}p = p(1 + 2 + 2^2 + \ldots + 2^{n-2})$$

$$= (2^{n-1} - 1)p$$

Now, add the two columns together to get:

$$p + p(2^{n-1} - 1) = p(1 + 2^{k-1} - 1)$$

$$= p(2^{n-1}) = N$$

Therefore, N is a perfect number.

Remark I: Every even perfect number N is of the form $N = 2^{n-1}(2^n - 1)$. This is known as Euler's Theorem, as well as the converse of Euclid's Theorem. So, just as Euclid's Theorem, this theorem is also true.

To get a better understanding of **Remark I**, let's demonstrate how it applies to the first four perfect numbers:

$$6 = (2)(3) = 2^1(2^2 - 1) = 2^{2-1}(2^2 - 1)$$

$$28 = (4)(7) = 2^2(2^3 - 1) = 2^{3-1}(2^3 - 1)$$

$$496 = (16)(31) = 2^4(2^5 - 1) = 2^{5-1}(2^5 - 1)$$

$$8128 = (64)(127) = 2^6(2^7 - 1) = 2^{7-1}(2^7 - 1)$$

Theorem II *The sum of the reciprocals of the factors of a perfect number N is equal to 2.*

Proof: Let $N = 2^{n-1}(2^n - 1)$, where $p = 2^n -$ And is prime. Then, list all possible factors of N.

Factorsof 2^{n-1}	Other Proper Factors
1	
2	$2p$
2^2	2^2p
.	.
.	.
.	.
2^{n-1}	$2^{n-1}p$

Now, calculate the sum of reciprocals for the factors of 2^{n-1}:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \ldots + \frac{1}{2^{n-1}}$$

$$= \frac{2^{n-1}}{2^{n-1}} + \frac{2^{n-1}}{2(2^{n-1})} + \frac{2^{n-1}}{2^2(2^{n-1})} \ldots + \frac{1}{(2^{n-1})}$$

$$=\frac{2^{n-1}}{2^{n-1}}+\frac{2^{n-1}\cdot 2^{-1}}{2^{n-1}}+\frac{2^{n-1}\cdot 2^{-2}}{2^{n-1}}\ldots+\frac{1}{2^{n-1}}$$

$$=\frac{2^{n-1}}{2^{n-1}}+\frac{2^{n-2}}{2^{n-1}}+\frac{2^{n-3}}{2^{n-1}}\ldots+\frac{1}{2^{n-1}}$$

$$=\frac{2^{n-1}+2^{n-2}+2^{n-3}\ldots+1}{2^{n-1}}$$

$$=\frac{2^{n}-1}{2^{n-1}}=\frac{p}{2^{n-1}}$$

Calculate the sum of reciprocals for the other factors:

$$\frac{1}{p}+\frac{1}{2p}+\frac{1}{2^{2}p}\ldots+\frac{1}{2^{n-1}p}$$

$$=\frac{2^{n-1}}{2^{n-1}p}+\frac{2^{n-1}}{2(2^{n-1}p)}+\frac{2^{n-1}}{2^{2}(2^{n-1}p)}\ldots+\frac{1}{(2^{n-1}p)}$$

$$=\frac{2^{n-1}}{2^{n-1}p}+\frac{2^{n-1}\cdot 2^{-1}}{2^{n-1}p}+\frac{2^{n-1}\cdot 2^{-2}}{2^{n-1}p}\ldots+\frac{1}{2^{n-1}p}$$

$$=\frac{2^{n-1}}{2^{n-1}p}+\frac{2^{n-2}}{2^{n-1}p}+\frac{2^{n-3}}{2^{n-1}p}\ldots+\frac{1}{2^{n-1}p}$$

$$=\frac{2^{n-1}+2^{n-2}+2^{n-3}\ldots+1}{2^{n-1}p}$$

$$=\frac{2^{n-1}}{2^{n-1}p}=\frac{p}{2^{n-1}p}=\frac{1}{2^{n-1}}$$

Then calculate the sum of reciprocals for all factors:

$$\frac{p}{2^{n-1}}+\frac{1}{2^{n-1}}$$

$$=\frac{p+1}{2^{n-1}} = \frac{2^n-1+1}{2^{n-1}} = \frac{2^n}{2^{n-1}} = 2$$

Application: The factors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, 248, and 496. So,

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{31}+\frac{1}{62}+\frac{1}{124}+\frac{1}{248}+\frac{1}{496}$$

$$=\frac{496+248+124+62+31+16+8+4+2+1}{496} = \frac{992}{496} = 2$$

Corollary I. No power of a prime can be a perfect number.

Proof. Let be prime, $n=p^k$, and $n=1, p, p^2, \ldots, pk$.

Now, $1+\frac{1}{p}+\frac{1}{p^2}+\ldots+\frac{1}{p^k}$

$$= 1+\frac{p^{k-1}+p^{k-2}+\ldots+p+1}{p^k}$$

$$=1+\frac{p^k-1}{p^k(p-1)} \leq 1+\frac{p^k-1}{p^k} = 1+\frac{p^k}{p^k}-\frac{1}{p^k}$$

$$= 1+1-\frac{1}{p^k}$$

$$= 2-\frac{1}{p^k} < 2$$

Hence, is not a perfect number.

Theorem III. *If N is perfect number such that* $N=2^{n-1}(2^n-1)$*, then the product of the positive divisors of N is equal to* N^n.

Proof. List the factors of *N*.

Factorsof2^{n-1}	Other Proper Factors
1	
2	$2p$
2^2	2^2p
.	.
.	.
.	.
2^{n-1}	$2^{n-1}p$

The product of the first column:

$$1\cdot2\cdot2^2\cdot2^3\cdot\ldots\cdot2^{n-1}=2^{1+2+3\ldots+(n+1)}=2^{\frac{n(n-1)}{2}}$$

The product of the second column:

$$p\cdot2p\cdot2^2p\ldots\cdot2^{n-1}p$$

$$=p^n(1\cdot2\cdot2^2\ldots\cdot2^{n-1})=p^n(2^{\frac{n(n-1)}{2}})$$

Then the product of both columns is:

$$(2^{\frac{n(n-1)}{2}})(p^n)(2^{\frac{n(n-1)}{2}})$$

$$=(2^{n(n-1)})(p^n)=((2^{n-1})(p))^n=N^n$$

Example 1:

$N = 28 = 2^2(2^3 - 1)$

Factors are 1, 2, 4, 7, 14, 28

$\cdot\, 2\cdot4\cdot7\cdot14\cdot28=28\cdot28\cdot28=28^3$

Example 2: *N*

$N = 6 = 2(2^2 - 1)$

Factors are 1, 2, 3, 6

$\cdot\, 2\cdot3\cdot6 = 6\cdot6 = 6^2$

Theorem IV. *Every even perfect number N is a triangular number.*

Definition: Triangular numbers are formed by the partial sum of the series of natural numbers and may be written as $T_n = 1 + 2 + 3 + 4 + \ldots + n$.

A visual:

$T_1 =$
$T_2 = 1 + 2 = 3$
$T_2 = 1 + 2 + 3 = 6$
$T_2 = 1 + 2 + 3 + 4 = 10$

And so on.
This sequence shows up in many places of mathematics. The triangular number T_x is a figurate

number that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one, as shown below.

					*
				*	**
			*	**	***
		*	**	***	****
	*	**	***	****	*****
*	**	***	****	*****	******
1	3	6	10	15	21

.**Corollary.** If T is a perfect number, then 8T+1 is a perfect square.

Proof. T is a perfect number T is a triangular number where $T = \frac{(m+1)m}{2}$ for some

Positive integer m. Now, $8T + 1 = 4m\,(m + 1) +$

$$= 4m^2 + 4m +$$

$$= (2m + 1)^2$$

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