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Research Paper

« The Addition and Subtraction Operations for two Continued Fractions »

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Abstract:
The continued fractions (C.F.) with positive non integer numerators are considered. The addition and subtraction operations of two continued fractions are discovered. Many definitions and examples that we used of these low are presented.
Keywords: Continued fractions, Addition operation of continued fractions, Subtraction operation of continued fractions.

1. INTRODUCTION

Continued fractions have been extensively studied and there is a large body of research related to them [3]. They are now included as classical results in most undergraduate textbooks like [1, 4, 5] and they continue to be studied a great deal. Continued fraction expansions can also be finite or infinite. For example, any rational number can be expressed as a finite continued fraction expansion, while an irrational number has an infinite continued fraction expansion [2, 9]. Although our main concentration is in the operations of the continued fractions, it is important to look for similarities. Previous studies have shown that the operations of the simple continued fractions follow similar patterns to those of continued fractions under certain circumstances [6, 7, and 8]. Therefore it is important to make this comparison. Hence, we summarize some important results for simple continued fractions. The main justification for the paper is the operations of the continued fractions. The addition and subtraction of two continued fractions.

II. PRELIMINARIES

Definition 2.1: A continued fraction is an expression of the form $a_0+\cfrac{b_0}{a_1+\cfrac{b_1}{a_2+\cfrac{b_2}{\ddots+\cfrac{b_n}{a_n}}}}$ where $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ can be real or complex, and their numbers can be either finite or infinite. In this paper, we consider the continued fractions of the form $a_0+\cfrac{z}{a_1+\cfrac{z}{a_2+\cfrac{z}{\ddots+\cfrac{z}{a_n}}}}$, where a_0 is an integer, $a_1, a_2, a_3, \ldots, a_n$ are positive integer and z is a positive non integer. We denote it by $[a_0\,;a_1\,,a_2\,,\ldots]_z$.

Example 2.1

(a) $3+\cfrac{\frac{3}{2}}{4+\cfrac{\frac{3}{2}}{7}}=[3;4,7]_{\frac{3}{2}}$. (b) $-7+\cfrac{\frac{2}{3}}{1+\cfrac{\frac{2}{3}}{3}}=[-7;1,3]_{\frac{2}{3}}$. (c) $1+\cfrac{\frac{5}{3}}{2+\cfrac{\frac{5}{3}}{3+\cfrac{\frac{5}{3}}{\cdots}}}=[1;2,3,\ldots]_{\frac{5}{3}}$

$(a), (b)$ have a finite number of terms, and (c) has an infinite number of terms.

Theorem 2.1: A number is rational if and only if it can be expressed as a finite C.F. [8].

Example 2.2:

(a) $\frac{40}{11} = 3 + \frac{\frac{3}{2}}{\frac{33}{14}} = 3 + \frac{\frac{3}{2}}{2+\frac{\frac{3}{2}}{\frac{21}{5}}} = 3 + \frac{\frac{3}{2}}{2+\frac{\frac{3}{2}}{4+\frac{\frac{3}{2}}{\frac{15}{2}}}} = 3 + \frac{\frac{3}{2}}{2+\frac{\frac{3}{2}}{4+\frac{\frac{3}{2}}{7+\frac{\frac{3}{2}}{3}}}} = [3,2,4,7,3]_{3/2}$.

(b) $\frac{28}{11} = 2 + \frac{\frac{2}{3}}{\frac{11}{9}} = 2 + \frac{\frac{2}{3}}{1+\frac{\frac{2}{3}}{3}} = [2\,,1,3]_{2/3}$.

(c) $\frac{3}{7} = 0 + \frac{\frac{3}{2}}{\frac{7}{2}} = 0 + \frac{\frac{3}{2}}{3+\frac{\frac{3}{2}}{3}} = [0,3,3]_{3/2}$.

(d) $\frac{7}{9} = 0 + \frac{\frac{7}{9}}{1} = [0;1]_{7/9}$.

Remark 2.1: To expand a negative rational number $-\frac{a}{b}(a\,,b > 0)$ into C.F we take the greatest integer number $\left\lfloor\frac{-a}{b}\right\rfloor$ for the first term of C.F that is, $\left\lfloor\frac{-a}{b}\right\rfloor = a'_0$, where $-a'_0 \leq -\frac{a}{b} < -a'_1 + 1$. We write $-\frac{a}{b} = -a'_0 + \frac{z}{\frac{a'}{b'}}$ and then we use the same techniques in theorem 2.1 to get the remaining terms for $\frac{a'}{b'}$. That is, if $\frac{a'}{b'} = [a'_1, a'_2 \cdots, a'_n]_z$. Then $\frac{-a}{b} = [a'_0; a'_1, a'_2 \cdots, a'_n]_z$. [6].

Example 2.3: $-\frac{71}{11} = -7 + \frac{6}{11} = -7 + \frac{\frac{2}{3}}{\frac{11}{9}} = -7 + \frac{\frac{2}{3}}{1+\frac{\frac{2}{3}}{3}} = [-7,1,3]_{2/3}$.

Lemma 2.1: $[c_0;c_1,\dots,c_{j-1},0,c_{j+1},\dots,c_n]_z = [c_0;c_1,\dots,c_{j-2},c_{j-1}+c_{j+1},c_{j+2},\dots,c_n]_z$

III. ADDITION OPERATION FOR TWO CONTINUED FRACTIONS

Definition 3.1: The C.F $[a_0\,;a_1\,,a_2\,,\dots,a_n]_z$ can be defined by $[a_0;a_1,a_2,\dots,a_n]_z = a_0 + \frac{z\,k_{n-1}(a_2)}{k_n(a_1)}$ or $[a_0\,;a_1\,,a_2\,,\dots,a_n]_z = \frac{k_{n+1}(a_0)}{k_n(a_1)}$, where $k_i(a_j) = a_{i+(j-1)}k_{i-1}(a_j) + zk_{i-2}(a_j)$, $i = 1,2,\dots,n$, $j = 0\,,1\,,\dots,n$ and $k_{-i}(a_j) = 0, k_0(a_j) = 1$.

Definition 3.2: Let $[a_0;a_1,\dots,a_m]_z$ and $[b_0;b_1,\dots,b_n]_z$ be two C.F. we define addition by

(1) If $m = n$ then $[a_0;a_1,\dots,a_n]_z + [b_0;b_1,\dots,b_n]_z = [c_0;c_1,\dots,c_n]_z$ (3.2a)

where

$c_o = a_0 + b_o = k_1(a_0) + k_1(b_0)$.

$c_1 = \left\lfloor\frac{a_1b_1}{a_1+b_1}\right\rfloor = \left\lfloor\frac{k_1(a_1)k_1(b_1)}{k_1(a_1)+k_1(b_1)}\right\rfloor$.

In general

$$c_m = \begin{cases} \left\lfloor\dfrac{z\left(k_m(a_1)k_m(b_1)k_{m-3}(c_2)-\left(k_{m-1}(a_2)k_m(b_1)+k_{m-1}(b_2)k_m(a_1)\right)k_{m-2}(c_1)\right)}{\left(k_{m-1}(a_2)k_m(b_1)+k_{m-1}(b_2)k_m(a_1)\right)k_{m-1}(c_1)-k_m(a_1)k_m(b_1)k_{m-2}(c_2)}\right\rfloor, m\ odd \\ \left\lfloor\dfrac{z\left(\left(k_{m-1}(a_2)k_m(b_1)+k_{m-1}(b_2)k_m(a_1)\right)k_{m-2}(c_1)-k_m(a_1)k_m(b_1)k_{m-3}(c_2)\right)}{k_m(a_1)k_m(b_1)k_{m-2}(c_2)-\left(k_{m-1}(a_2)k_m(b_1)+k_{m-1}(b_2)k_m(a_1)\right)k_{m-1}(c_1)}\right\rfloor, m\ even \end{cases},$$

for $m = 2,\dots,n$. The last term c_n of the resulting C.F. is to be expanded again as a C.F.

(2) If $m \neq n$ (suppose that $m < n$) then

$[a_0;a_1,\dots,a_m]_z + [b_0;b_1,\dots,b_m,b_{m+1},\dots,b_n]_z = [c'_0;c'_1,\dots,c'_m,c'_{m+1},\dots,c'_n]_z$ (3.2b),

where $c'_j = c_j$ for $j = 0,1,2,\dots,m$ (c_j as we did for case $m = n$), while $c'_j = c_{j\,,j-m}$ and

$$c_{j,j-m} = \begin{cases} \left\lfloor\dfrac{z\left(k_m(a_1)k_j(b_1)k_{j-3}(c'_2)-\left(k_{m-1}(a_2)k_j(b_1)+k_{j-1}(b_2)k_m(a_1)\right)k_{j-2}(c'_1)\right)}{\left(k_{m-1}(a_2)k_j(b_1)+k_{j-1}(b_2)k_m(a_1)\right)k_{j-1}(c'_1)-k_m(a_1)k_j(b_1)k_{j-2}(c'_2)}\right\rfloor; j\ odd \\ \left\lfloor\dfrac{z\left(\left(k_{m-1}(a_2)k_j(b_1)+k_{j-1}(b_2)k_m(a_1)\right)k_{j-2}(c'_1)-k_m(a_1)k_j(b_1)k_{j-3}(c'_2)\right)}{k_m(a_1)k_j(b_1)k_{j-2}(c'_2)-\left(k_{m-1}(a_2)k_j(b_1)+k_{j-1}(b_2)k_m(a_1)\right)k_{j-1}(c'_1)}\right\rfloor; j\ even \end{cases},$$

for $j = m+1, m+2, \cdots, n$. Also, the last term of the resulting C.F is to be treated as mentioned earlier.

Example 3.1: Find $[2;1,3]_{2/3} + [3;1,14]_{2/3}$

Solution: Let $[2;1,3]_{2/3} = [a_0;a_1,a_2]_z$ and $[3;1,14]_{2/3} = [b_0;b_1,b_2]_z$. We get $m = n = 2$. From definition (3.2a) we get $[a_0;a_1,a_2]_z + [b_0;b_1,b_2]_z = [c_0;c_1,c_2]_z$, where c_2 is the last term and

$c_0 = a_0 + b_0 = 5$.

$c_1 = \left\lfloor\frac{a_1b_1}{a_1+b_1}\right\rfloor = \left\lfloor\frac{1}{2}\right\rfloor = 0$.

$c_2 = \frac{z\left(a_2(b_1b_2+z)+b_2(a_1a_2+z)\right)}{(a_1a_2+z)(b_1b_2+z)(1)-\left(a_2(b_1b_2+z)+b_2(a_1a_2+z)\right)c_1} = \frac{13}{11}$ (to be treated as C. F),

$= [1;3,1]_{2/3}$.

Then $[2;1,3]_{2/3} + [3;1,14]_{2/3} = [5;0,1,3,1]_{2/3}$.

Example 3.2: Find$[2;2]_{7/4} + [2;9,5.7]_{7/4}$.

Solution: Let $[2;2]_{7/4} = [a_0;a_1]_{7/4}$ and $[2;9,5.7]_{7/4} = [b_0;b_1,b_2,b_3]_{7/4}$, we get $m = 1$, $n = 3$ and $m < n$, from definition (3.2b) we have $[2;2]_{7/4} + [2;9,5.7]_{7/4} = [a_0;a_1]_{7/4} + [b_0;b_1,b_2,b_3]_{7/4} = [c'_0;c'_1,c'_2,c'_3]_z$, where c'_3 is last term and

$c'_0 = c_0 = a_0 + b_0 = 2 + 2 = 4.$

$c'_1 = c_1 = \left\lfloor \frac{a_1 b_1}{a_1+b_1} \right\rfloor = \left\lfloor \frac{2\cdot 9}{2+9} \right\rfloor = \left\lfloor \frac{18}{11} \right\rfloor = 1.$

$$c'_2 = c_{2,1} = \left\lfloor \frac{z\left(\left(k_0(a_2)k_2(b_1)+k_1(b_2)k_1(a_1)\right)k_0(c'_1)-k_1(a_1)k_2(b_1)k_{-1}(c'_2)\right)}{k_1(a_1)k_2(b_1)k_0(c'_2)-\left(k_0(a_2)k_2(b_1)+k_1(b_2)k_1(a_1)k_1(c'_1)\right)} \right\rfloor$$

$$= \left\lfloor \frac{z\left(\left((1)(b_1b_2+z)+b_2a_1\right)(1)-a_1(b_1b_2+z)(0)\right)}{a_1(b_1b_2+z)(1)-\left((1)(b_1b_2+z)+b_2a_1\right)c'_1} \right\rfloor = \left\lfloor \frac{1589}{588} \right\rfloor = 2.$$

$$c'_3 = c_{3,2} = \frac{z\left(k_1(a_1)k_3(b_1)k_0(c'_2)-\left(k_0(a_2)k_3(b_1)+k_2(b_2)k_1(a_1)\right)k_1(c'_1)\right)}{\left(k_0(a_2)k_3(b_1)+k_2(b_2)k_1(a_1)\right)k_2(c'_1)-k_1(a_1)k_3(b_1)k_1(c'_2)} = \frac{z\left(a_1(b_1b_2b_3+zb_1+zb_3)(1)-\left((1)(b_1b_2b_3+zb_1+zb_3)+(b_2b_3+z)a_1\right)c'_1\right)}{\left((1)(b_1b_2b_3+zb_1+zb_3)+(b_2b_3+z)a_1\right)\left(c'_1c'_2+z\right)-a_1(b_1b_2b_3+zb_1+zb_3)c'_2} = \frac{77}{31}$$ (to be treated as C. F)

$= [2,3,2,2,19,2,4,5,2,3,8,4,4,2,2,2,9,5,7]_{7/4}$.

Then $[2;2]_{7/4} + [2;9,5,7]_{7/4} = [4;1,2,2,3,2,2,19,2,4,5,2,3,8,4,4,2,2,2,9,5,7]_{7/4}$.

Example 3.3: Find $[4]_{7/4} + [0;7,5]_{7/4}$

Solution: Let $[4]_{7/4} = [a_0]_z$ and $[0;7,5]_{7/4} = [b_0;b_1,b_2]_z$, we get m =0, $n = 2$ ($m < n$) and from definition (3.2b), we have $[4]_{7/4} + [0;7,5]_{7/4} = [a_0]_z + [b_0;b_1,b_2]_{7/4} = [c_0;b_1,b_2]_{7/4}$, where

$c_0 = a_0 + b_0 = 4 + 0 = 4.$

Then$[4]_{7/4} + [0;7,5]_{7/4} = [4;7,5]_{7/4}$.

IV. SUBTRACTION OPERATION FOR TWO CONTINUED FRACTIONS

Definition 4.1: Let $[a_0;a_1,\cdots,a_n]_z$ be a C.F then we can define additive inverses by $-[a_0;a_1,\cdots,a_n]_z = [b'_0;b'_n]_z$, where

$b'_0 = -1 - a_0 = f_0.$

$b'_n = \frac{zk_n(a_1)}{k_n(a_1)-zk_{n-1}(a_2)}$ (to be treated as C. F.),

$= [f_1;f_2,\cdots,f_l]_z.$

Therefore $-[a_0;a_1,\cdots,a_n]_z = [b'_0;b'_n]_z = [f_0;f_1,\cdots,f_l]_z$.

Example 4.1: Find $-[2;1,3]_{2/3}$.

Solution: Let $[2;1,3]_{2/3} = [a_0;a_1,a_2]_z$, ($n = 2$) from definition 4.1, we have: $[-2;1,3]_{2/3} = [b'_0;b'_2]_z$, where

$b'_0 = -1 - a_0 = -1 - 2 = -3.$

$b'_2 = \frac{zk_2(a_1)}{k_2(a_1)-zk_1(a_2)} = \frac{22}{15}$ (to be treated as C. F).

$= [1;1,1,1,3,2]_{2/3}$.

Then $-[2;1,3]_{2/3} = [-3;1,1,1,1,3,2]_{2/3}$

Definition 4.2: If $[a_0;a_1,\cdots,a_m]_z$ and $[b_0;b_1,\cdots,b_n]_z$ are two C.F then we define subtraction $[a_0;a_1,\cdots,a_m]_z - [b_0;b_1,\cdots,b_n]_z$ by the addition $[a_o;a_1,\cdots,a_m]_z + [f_0;f_1.,\cdots,f_l]_z$ where $[f_0;f_1,\cdots,f_l]_z$ give by definition 4.1. That is

(1) If $m = l$ then $[a_o;a_1,\cdots,a_m]_z + [f_0;f_1,\cdots,f_m]_z = [c_0;c_1,\cdots,c_m]_z$ (4.2a)

where $c_0, c_1, \cdots, c_m$ as give in (3.2a)

(2) If $m \neq l$,($l < m$) then

$[a_0;a_1,\cdots,a_l,a_{l+1},\cdots,a_m]_z + [f_0;f_1,\cdots,f_l]_z = [c'_0;c'_1,\cdots,c'_l,c'_{l+1},\cdots,c'_m]_z$ (4.2b)

where $c'_0, c'_1, \cdots, c'_m$ as give in (3.2b) .

Example 4.2: Find $[1;1,2]_{5/3} - [0;1,2]_{5/3}$

Solution: (1) first we find $-[0;1,2]_{5/3}$.Let $[0;1,2]_{5/3} = [a_0;a_1,a_2]_{5/3}$, ($n = 2$). From definition 4.1, we have $-[0;1,2]_{5/3} = [b'_0, b'_2]_{5/3}$, where

$b'_0 = -1 - a_0 = -1 - 0 = -1$

$$b'_2 = \frac{zk_2(a_1)}{k_2(a_1) - zk_1(a_2)} = \frac{z(a_1a_2 + z)}{(a_1a_2 + z) - z(a_2)} = \frac{55}{3} = [18;5]_{5/3}.$$

Then $-[0;1,2]_{5/3} = [-1;18,5]_{5/3}$.

(2) To find $[1;1,2]_{5/3} + [-1;18,5]_{5/3}$, we let $[1;1,2]_{5/3} = [a_0;a_1,a_2]_z$ and $[-1,18,5]_{5/3} = [f_0;f_1,f_2]_z$, ($m = n = 2$) from (3.2a), we have

$[1;1,2]_{5/3} + [-1;18,5]_{5/3} = [a_0;a_1,a_2]_z + [f_0;f_1,f_2]_z = [c_0;c_1,c_2]_z$, where c_2 is the last term and

$c_0 = a_0 + f_0 = 1 + (-1) = 0,$

$c_1 = \left\lfloor \frac{a_1 f_1}{a_1+f_1} \right\rfloor = \left\lfloor \frac{1\cdot 18}{1+18} \right\rfloor = \left\lfloor \frac{18}{19} \right\rfloor = 0,$

$c_2 = \frac{z(a_2(f_1f_2+z)+f_2(a_1a_2+z))}{(a_1a_2+z)(f_1f_2+z)-(a_2(f_1f_2+z)+f_2(a_1a_2+z))c_1} = \frac{3025}{3025} = 1,$
Then $[1;1,2]_{5/_3} + [-1;18,5]_{5/_3} = [0;0,1]_{5/_3} = 1$ (by lemma 2.1)

V. CONCLUSION

This paper is a sequel to our previous work in which we found the operations of the simple continued fractions. In the previous work, [8] we discovered the definitions of addition, subtractions, multiplication and division operations of the simple continued fractions. In this paper we defined the addition and subtractions of continued fractions with positive non Integer numerators. The study of the continued fractions with positive non Integer numerators is the first step to understand the continued fractions with polynomial numerators.

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