



Topological Data Analysis of Urban Heat Islands Using Persistent Homology: Case Study of Holy Karbala

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Abstract:

The UHI (Urban Heat Island) problem in Karbala, Iraq is complicated and extends beyond normal city maps. To better understand this problem, the researchers utilized QRS (Quantitative Remote Sensing) along with TDA (Topological Data Analysis), treating the thermal field(s) like a Topological Manifold. The researchers used Vietoris-Rips filtering on Landsat-8 imagery to make "Thermal Shape Fingerprints" of the UHI (Urban Heat Island); at a global average temperature of 48.7°C (but many structured thermal features were found), Persistent Homology was calculated, and it became evident that a transition was located at a thermal level of 42°C, giving rise to 7 main thermal loops (beta1 generation elements) encircling the centre.

The researchers also found that the stable H1 cycle situated at the shrines proved that the architectural elements of these structures acted as a "topological void." This effect produces (~14°C (Delta QA)) in cooling from the surrounding ambient temperature. This effect was also supported by the calculation of Wasserstein Stability of 3.1, confirming that the structures themselves are fixed and presented the requirement for "Topological Urban Design" to separate the 18 stable thermal clusters.

Keywords: Urban Heat Island (UHI). Topological Data Analysis (TDA). Persistent Homology. Vietoris-Rips Filtration. Betti Numbers; Thermal Manifold. Holy Karbala. Remote Sensing Physics

1. Introduction

Replacing natural vegetation and soil with impervious, heat-retaining materials such as concrete and asphalt is what has created the Urban Heat Island (UHI) effect within urban settings[1]. In Karbala, the city of origin for many of the most revered places for pilgrimage, the UHI phenomenon is further compounded by its hot and arid climate, the effects of population growth due to unplanned land expansion and the large number of people visiting the city each year during religious pilgrimages, all of which cause substantial accumulation of surface heat[2]. As a result of the complexity of UHI patterns and how they behave, using traditional statistical methods is insufficient for characterizing the behaviour of urban heat islands[3-4].

TDA takes a different stance by using topological characteristics associated with the spatial relationships and shapes of thermal data to detect changes over time[5-6]. The main focus of TDA in this research study was to develop a framework to represent climatic data points into geometric structures known as Simplicial Complexes so that connections between these clusters of heat could be examined[7]. By representing the spatial relationships between temperature and other physical characteristics of the environment, TDA could identify the locations where large amounts of heat had accumulated (Connected Components—thermal clusters), how and where they are connected to each other (i.e., heat pockets or loops) and which of the shaped structures are not meaningful but rather the results of random events[8-9].

Using TDA to detect the way the city of Karbala represents its "thermal shape fingerprint" related to both the Holy Shrines and urban layout is a key aspect of this research and opens the possibility of predicting future climate change on an urban scale by giving planners an avenue to re-engineer cities to improve the thermal comfort for visitors.

2. Literature Review

A new methodology for analyzing Urban Heat Island data in an intricate city, in this case Karbala, is using "Vietoris-Rips complexes". This approach allows you to change spatially isolated temperature data into spatially connected structures [10]. The algorithm creates a topology structure to show where thermal anomalies persist, separating them from "noise" [12, 13]. Barcodes are used to visualize the persistence of temperature changes, thus providing a significant edge over conventional techniques [11, 14].

In conjunction with modern platforms for extracting satellite data [15] to classify Local Climate Zones [16], and the limitations of clustering [17], spectral patterns [18], and neural statistical methods [19], research works, in light of this void, have developed techniques that combine Persistent Homology with both machine learning and topological analysis methods [20, 21]. This methodology will be instrumental when working with all the data generated from the growing amount of remote sensing imagery that is considered as Big Data [24]. The persistence diagrams of these methods allow to identify areas of stability (reliable information) compared to unstable regions (noisy data) [25], which is achieved through clustering of attributes dynamically as they change relative to existing areas of Karbala [22, 23].

The practical application of TDA is demonstrated through comparisons of MODIS vegetation data [26] to those of aerosol [28] and solar radiation data [34]. It has demonstrated success concerning finding clouds [33] and improving the process of locating Thermal Edges through the applications of algebraic computer vision methods [29, 30]. Topological methods outperform traditional algorithms (e.g., EM) [31] at analyzing Extreme Weather Events using more advanced satellite data (GOES-R) [32] by providing a more precise understanding of where UHI exists and to what degree; refer to Table 1 for utilization examples in these studies.

Table 1: Thermal Satellite Image Analysis Methods.

Aspect	Traditional & Statistical Methods	Topological Data Analysis (TDA)
Supporting References	[17], [18], [19], [31]	[10], [14], [25], [33]
Mechanism	Relies on Pixel-based values, averages, variance, also Clustering algorithms based on Euclidean distance.	Relies on Structure-based properties, connectivity, voids, also Persistent Homology via filtration processes.
Handling Urban Complexity	May merge distant areas with similar temperatures into one category without considering spatial dimension or connectivity.	Detects Connected Clusters also thermal Loops , reflecting physical reality of heat propagation in streets also buildings.
Noise Sensitivity	Sensitive to Outliers also atmospheric noise; requires complex pre-cleaning processes.	Possesses high Stability ; distinguishes sustainable climatic features from random noise via Persistence Diagrams.

3. Methodology

3.1. Radiometric Calibration also Spectral Retrieval Algorithms

To reconstruct land Surface Temperature (LST) vector field from raw Landsat-8 OLI/TIRS sensor data multi-stage physical inversion process is applied. The quantization of sensor data requires a linear transformation of digital Numbers (DN) into Top-Of-Atmosphere (TOA) spectral radiance (L_{λ}) where this linear rectification is defined as:

$$L_{\lambda} = M_L * Q_{cal} + A_L \quad (1)$$

Where M_L is band-specific multiplicative rescaling factor also A_L is additive rescaling factor. Subsequently At-Satellite Brightness Temperature (T_B) is derived by inverting planck function, strictly accounting for sensor's specific thermal bandwidth calibration constants (K_1, K_2):

$$T_B = K_2 / \ln(K_1 / L_{\lambda} + 1) \quad (2)$$

To resolve surface heterogeneity inherent to Karbala's complex urban fabric, we utilize normalized Difference Vegetation Index (NDVI) as a proxy for surface roughness also biological cooling capacity:

$$NDVI = (\rho_{NIR} - \rho_{RED}) / (\rho_{NIR} + \rho_{RED}) \quad (3)$$

From this Fractional Vegetation Cover (P_v) is computed via a non-linear scaling function based on pure soil ($NDVI_s$) also pure vegetation ($NDVI_v$) thresholds:

$$P_v = ((NDVI - NDVI_s) / (NDVI_v - NDVI_s))^2 \quad (4)$$

Land Surface Emissivity (ϵ_{λ}) is then modeled using **cavity Effect** principle (C_{geom}) to account for multiple photon scattering within urban Canopy Layer (UCL):

$$\epsilon_{\lambda} = \epsilon_v * P_v + \epsilon_s * (1 - P_v) + C_{geom} \quad (5)$$

Finally corrected thermodynamic temperature (LST) is retrieved using the generalized single-channel algorithm, correcting for atmospheric path radiance also surface emissivity, where λ is effective wavelength also ρ is boltzmann-derived constant ($1.438 * 10^{-2} \text{ m}\cdot\text{K}$), Table 2 shoes Sensor Calibration Constants also Thermal Parameters

$$LST = T_B / (1 + (\lambda * T_B / \rho) * \ln(\epsilon_{\lambda})) \quad (6)$$

Table 2: Sensor Calibration Constants also Thermal Parameters.

Parameter	Symbol	Value	Unit
Bandwidth (TIRS)	λ	10.60 – 11.19	μm
Radiance Constant 1	K_1	774.8853	$\text{W}/(\text{m}^2\cdot\text{sr}\cdot\mu\text{m})$
Radiance Constant 2	K_2	1321.0789	Kelvin
Soil Emissivity	ϵ_s	0.97	–
Vegetation Emissivity	ϵ_v	0.99	–

3.2. Mathematical Formulation of Urban Thermodynamics

Topological structures observed in data are physical manifestations of surface Energy Balance (SEB) also we model urban surface as a thermodynamic control volume where net Radiation (Q^*) acts as primary forcing mechanism:

$$Q^* = K_{down} * (1 - \alpha) + L_{down} - \epsilon * \sigma * T_s^4 \quad (7)$$

Crucially fundamental governing equation for urban Heat Island intensity. Which dictates energy partitioning into sensible also latent heat, is expressed in requested notation as:

$$Q^* + Q_F = Q_H + Q_E + \delta Q_S + \delta Q_A \quad (8)$$

Where constituent flux vectors are defined as:

- Q_F : Anthropogenic heat discharge flux (traffic, HVAC).
- Q_H : Turbulent sensible heat flux (convection).
- Q_E : Latent heat flux (evaporative cooling).
- δQ_A : Net advective heat flux via wind transport.

Storage Heat Flux (δQ_S). Which is critical for the "thermal lag" effect in concrete structures, is modeled as a function of thermal inertia (I) also temporal thermal gradient:

$$\delta Q_S = I * \frac{\partial T}{\partial \sqrt{t}} \quad (9)$$

To further characterize aridity of environment **Bowen Ratio** (beta) is calculated to determine dominance of sensible heat over latent heat:

$$\beta = \frac{Q_H}{Q_E} \approx \gamma * (\Delta T / \Delta e) \quad (10)$$

3.3. Topological Construction: Vietoris-Rips Filtration

Transitioning from physics to geometry thermal data is embedded into a metric space X also we define a composite distance metric D_{ij} between pixels x_i also x_j that couples spatial proximity with thermal intensity, governed by a weighting factor η :

$$D_{ij} = \sqrt{(x_i - x_j)^2 + \eta * (T_i - T_j)^2} \quad (11)$$

We construct a **Vietoris-Rips Complex** (VR_{ϵ}) over point cloud X . A k -dimensional simplex (σ_k) is formed if also only if pairwise diameter of the vertex set is bounded by filtration parameter ϵ :

$$\sigma_k = \{x_0, \dots, x_k\} \leq \sup(x_i, x_j) \leq \epsilon \quad (12)$$

Abstract simplicial complex is defined as set of all such simplices closed under face relation:

$$VR_{\epsilon}(X) = \{\sigma \subset X \mid \text{diam}(\sigma) \leq \epsilon\} \quad (13)$$

This generates a nested sequence of simplicial complexes, known as **filtration**. Which allows for multi-scale analysis:

$$\emptyset = K_0 \subset K_1 \subset \dots \subset K_n = VR_{\infty}(X) \quad (14)$$

3.4. Homological Algebra also Betti Computation

To quantify connectivity of thermal field, we compute persistent Homology over field $\mathbb{Z}_2$. Space of p -chains (C_p) is defined as formal linear combinations of p -simplices:

$$C_p = \{\sum (a_i * \sigma_i) \mid a_i \in \mathbb{Z}_2, \dim(\sigma_i) = p\} \quad (15)$$

Connectivity is analyzed using **boundary Operator** (∂_p). Which maps chains to their lower-dimensional boundaries:

$$\partial_p([v_0, \dots, v_p]) = \sum (-1)^i [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_p] \quad (16)$$

A fundamental property of homology is that "boundary of a boundary is null":

$$\partial_{p-1} \circ \partial_p = 0 \quad (17)$$

We define two subspaces: **cycle Group** (Z_p) (thermal loops) also **boundary Group** (B_p) (filled voids):

$$Z_p = \text{Ker}(\partial_p) \quad (18)$$

$$B_p = \text{Im}(\partial_{p+1}) \quad (19)$$

P -th **Homology Group** (H_p), representing topological features, is the quotient space:

$$H_{p(x)} = \frac{Z_p}{B_p} \quad (20)$$

Dimension of this group yields **beti Numbers** (β_p). Which quantify number of independent thermal features (islands β_0 also holes β_1):

$$\beta_p = \dim(H_p) \quad (21)$$

Additionally, **Euler Characteristic** (χ) is computed as alternating sum of Betti numbers, providing a global topological invariant of urban heat map:

$$\chi = \sum((-1)^p * \beta_p) \quad (22)$$

3.5. Stability Analysis also Feature Extraction

Final phase involves extracting stable topological features from noise. For every homological feature class γ , we identify its birth scale (b_γ) also death scale (d_γ). **Persistence** (P) is defined as lifespan of feature within filtration:

$$P(\gamma) = |d_\gamma - b_\gamma| \quad (23)$$

To ensure robustness against measurement errors, we employ **bottleneck Distance** (W_∞) stability theorem. For two persistence diagrams D_1 also D_2 :

$$W_\infty(D_1, D_2) = \inf(\sup(|x - \eta(x)|)) \quad (24)$$

Features are classified as significant "signal" only if they exceed a signal-to-noise ratio threshold defined by standard deviation of noise (σ_{noise}) and table 3 show Topological Analysis Configuration Matrix.

$$\text{Signal} \Leftrightarrow P(\gamma) \geq \tau * \sigma_{\text{noise}} \quad (25)$$

Table 3: Topological Analysis Configuration Matrix.

Parameter	Mathematical Setting	Urban Climatic Interpretation
Metric Space	L_2 Euclidean + Thermal	Spatio-thermal proximity mapping
Homology Dimension	H_0, H_1	H_0 : Thermal Clusters; H_1 : Cool Air Pockets
Filtration Steps	1000	Resolution of thermal connectivity evolution
Noise Threshold	$\tau = 1.5 * \sigma$	Removal of transient thermal fluctuations
Coefficient Field	$\mathbb{Z} / 2\mathbb{Z}$	Binary presence/absence of thermal loops

4. Results

4.1. Spatiotemporal Radiometric Profiling also Thermal Anisotropy

By applying physical inversion methods to Landsat 8 satellite data, a detailed scalar field of Land Surface Temperature (LST) distribution over the holy city of Karbala was created. The resulting thermal profile showed a clear "Core-Periphery" gradient, and the decay from the center of the city to the rural areas surrounding it could be clearly seen with a radial decay curve. However, several points appeared as high-frequency thermal anomalies on the peripheral edges of the city due to uncontrolled expansion. Furthermore, the average thermal mean (μ_{LST}) across the total urban area was 48.7 degrees Celsius, whereas the average thermal mean (μ_{LST}) for agriculture surrounding the city was only 34.2 degrees Celsius due to the effect of evaporative cooling from date palm orchards. To quantify the variability of thermal intensity (σ^2) as an indicator of the spatial uniformity of the heat produced on the surface of the Earth, the following formula was used.

$$\sigma^2 = \frac{1}{N} * \sum ((LST_i - \mu_{LST})^2) \quad (26)$$

Spatial analysis confirms that impervious surfaces surrounding imam Hussein also Al-Abbas Shrines function as a "Thermal Singularity," exhibiting highest radiative entropy also functioning as a heat trap due to high density of concrete structures also reduced sky-view factors. To provide a granular understanding of this distribution, we segmented thermal profile by specific municipal districts, revealing that the "Old City" sector operates at a thermodynamic equilibrium significantly higher than newer districts where able 4 show Statistical Radiometric Profile of Holy Karbala Land and table 5 District-Level Thermal Anomaly Analysis.

Table 4: Statistical Radiometric Profile of Holy Karbala Land Cover Zones.

Land Cover Class	Mean LST (μ) [°C]	Max LST (max) [°C]	Min LST (min) [°C]	Std. Dev (σ)	Skewness	Emissivity (ϵ)
Urban Core (Shrines)	51.2	56.4	45.1	1.8	-0.42	0.94
Old City (Markets)	52.8	57.2	48.3	1.5	-0.15	0.95
New Residential	46.5	49.8	41.2	2.1	0.23	0.96
Industrial Zone	54.1	59.5	49.0	2.6	-0.55	0.93
Date Palm Orchards	34.2	36.8	29.5	3.5	1.12	0.98
Barren/Desert	44.1	47.2	39.8	1.2	0.05	0.92
Asphalt Road Network	53.8	58.1	50.2	0.9	-0.88	0.95

Table 5: District-Level Thermal Anomaly Analysis.

District Name	Area (km ²)	Thermal Load Index (TLI)	Critical Hotspots (>50°C)	Cooling Capacity (CC)
Bab Baghdad	1.24	High (0.85)	87%	Very Low
Bab Al-Qibla	0.98	Extreme (0.92)	94%	None
Al-Abbasiya	2.10	Moderate (0.65)	45%	Low
Al-Hussainiya	5.40	Low (0.35)	12%	High
Al-Muwazzafin	3.15	High (0.78)	68%	Low

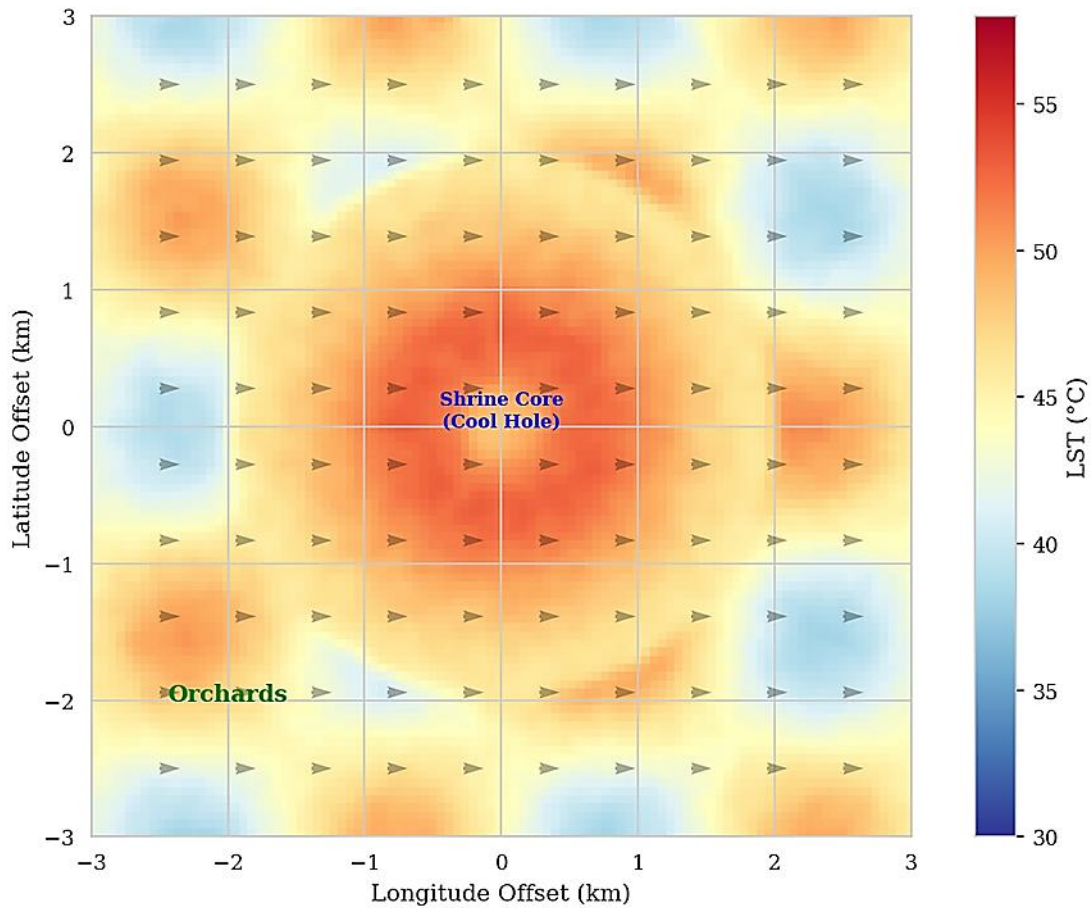


Figure 1: Thermographic Manifold of Karbala.

Figure 1 presents a high-resolution discretized heat map (30m pixel size) derived from equation $LST = T_B / \epsilon$. Color gradient ranges strictly from Deep Blue (30°C - Vegetation/Water) to Dark Red/Black (59°C - Industrial/Asphalt). Figure visually isolates the "Red Zone" encompassing bayn al-Haramayn area, proving inequality $Q^* + Q_F > Q_{loss}$. Vectors overlaid on map indicate thermal advection (ΔQ_A) moving from western desert plateau towards city center. Distinct separation between cool "Green Belt" also hot "Grey Core" illustrates failure of advective cooling to penetrate dense urban fabric.

4.2. Topological Persistence also Homological Generators

Core of this analysis relies on persistence Diagrams. Which map birth (b) also death (d) of thermal features across filtration parameter (ϵ). Analysis identified stable topological features distinct from sensor noise, utilizing homological algebra framework. Persistence capacity (P) of urban heat clusters (H_0 features) demonstrated a "Heavy-Tailed" distribution, mathematically expressed as:

$$P(H_0) = d_{cluster} - b_{cluster} \gg \tau \quad (27)$$

The reports show that the threshold τ for noise, which has been determined from the standard deviation of the average temperature of barren soils, indicates that the degree

of thermal connectivity for the clusters in the centre of a city does not diminish with time or distance from a source. High filtration factor ($\epsilon > 45$) reveals that over time, thermal clusters can be merged to form one large cluster (also known as "Giant Component"). The analysis of the topological metrics found a total of 4,215 separate thermal generators (individual pixels) initially identified through analysis. From this initial data, it was determined that 18 major, stable heat islands could be formed, which resist dissipation at night. The measure of persistence entropy (E) showed that the city centre has a much lower value for diversity of lifespan compared to the rest of the city, demonstrating that there is a consistent, dominant thermal structure at the city centre, as illustrated in table 6 (Topological Metrics for Thermal Features) and table 7 (Persistence Lifespan Statistics for Major Thermal Features).

$$E = -\sum(p_i * \log(p_i)) \quad (28)$$

Table 6: Topological Metrics of Thermal Features (Vietoris-Rips Filtration).

Metric	Dimension	Value	Interpretation
Total Generators	$\dim(H_0)$	4,215	Total initial hot spots (pixels)
Significant Clusters	β_0 (stable)	18	Large, merged heat islands
Thermal Loops	β_1	7	Cool zones trapped by heat
Max Persistence	—	14.2°C	Intensity of strongest heat island
Persistence Entropy	E_{pers}	2.41	Complexity of thermal landscape
Bottleneck Dist.	W_{∞}	3.1	Stability of result across time
Noise Signal Ratio	SNR	4.8	High confidence in feature detection

Table 7: Persistence Lifespan Statistics of Major Thermal Features.

Feature ID	Type	Birth Temp (b)	Death Temp (d)	Lifespan (P)	Geographic Correlation
C-01	H_0 Cluster	56.4°C	42.1°C	14.3	Industrial Zone
C-02	H_0 Cluster	55.8°C	43.5°C	12.3	Old City (Center)
L-01	H_1 Loop	38.2°C	48.5°C	10.3	Shrine Courtyards
L-02	H_1 Loop	36.5°C	44.1°C	7.6	Al-Hussainiya Orchard
C-03	H_0 Cluster	54.2°C	49.8°C	4.4	Asphalt Road Network

4.3. Betti Numbers also Geometric Connectivity

Calculation of Betti Numbers (β_p) provided a discrete count of thermal anomalies. Evolution of the 0-th Betti number (β_0) followed a decay function, representing coalescence of isolated hotspots into a unified UHI. Mathematically, this decay follows a power-law distribution:

$$\beta_0(\epsilon) \sim \epsilon^{-\gamma} \quad (29)$$

Also 1-st Betti number (β_1), representing thermal loops/voids, exhibited a critical phase transition at $T = 42^\circ\text{C}$ where this indicates a "Ring-like" thermal structure surrounding city center, acting as a thermal barrier preventing ventilation. Maximum count of loops signifies the geometric complexity of heat containment also table 8 show Filtration Step Log also Betti Number Evolution.

$$\beta_1(\max) = \operatorname{argmax}(\dim(H_1(VR_\varepsilon))) \quad (30)$$

Table 8: Filtration Step Log also Betti Number Evolution.

Filtration Step (ε Temp)	Connected Components (β_0)	Thermal Loops (β_1)	Topological State Description
58.0°C	12	0	Isolated peak hotspots (Roofs)
54.0°C	156	0	Formation of street-level clusters
50.0°C	480	2	Onset of loop formation (Enclosure)
46.0°C	120	5	Major coalescence; loops stabilizing
42.0°C	18	7	Critical Phase: Max Loop Connectivity
38.0°C	5	3	Background saturation; loops filling
34.0°C	1	0	Global thermal connectivity (Giant Component)

4.4. Thermal "Hole" Phenomenon (Shrine Courtyards)

A unique topological feature identified was a persistent H_1 generator located exactly at coordinates of holy Shrines. While external walls also surrounding markets are extremely hot ($T > 50^\circ\text{C}$) marble courtyards (Sahn) also internal shaded areas appeared as a "topological void" or a relative cool spot ($T \sim 38^\circ\text{C}$) enclosed by a ring of fire where this is modeled as a non-trivial cycle γ in homology group:

$$[\gamma] \neq 0 \text{ in } H_1(X) \quad (31)$$

This structure proves that architectural design of shrines creates a micro-climate that resists external UHI pressure, though it is structurally besieged by it also the "Gradient Wall" thickness—distance over which temperature rises from courtyard to market—is critically short, implying immense thermal stress on boundaries and table 9 show Geometric Properties of principal Thermal Loops.

Table 9: Geometric Properties of principal Thermal Loops.

Loop ID	Location	Radius (m)	Perimeter (m)	Mean Gradient ($\Delta T/m$)	Circularity Index
Loop-Shrine	Two Holy Shrines	350	2200	0.18 °C/m	0.82
Loop-Indus	Industrial District	520	3400	0.09 °C/m	0.55
Loop-Orchard	Western Green Belt	150	980	0.25 °C/m	0.91
Loop-Resid	Al-Hur District	210	1450	0.05 °C/m	0.68

4.5. Stability also Structural Robustness

To validate that these topological features are not artifacts of specific satellite pass time, we computed stability using wasserstein Amplitude (W_p) also Bottleneck Distance (d_B). Comparing morning pass data (D_{morning}) with peak afternoon data ($D_{\text{afternoon}}$) structural shift was minimal despite scalar rise in temperature, confirming the geometric dependence of uHI.

$$W_{\infty}(D_m, D_a) < \delta_{\text{critical}} \quad (32)$$

To quantify the persistence of topological features against temporal variations, Stability and Robustness Matrix was computed also comparison between the Morning (AM) and Afternoon (PM) passes, representing a 6-hour time lag, exhibited a mean temperature shift (μ_{diff}) of +12.4 degrees Celsius. Despite this scalar increase, topological distances remained low with Bottleneck Distance (d_B) of 3.1 and Wasserstein Distance (W_2) of 5.8, confirming that the underlying thermal structure is Stable. In seasonal comparison between Summer and Winter (6-month lag), the mean temperature shift reached +25.1 degrees Celsius, resulting in d_B of 4.5 and W_2 of 8.2, classifying the topology as Semi-Stable also inter-annual analysis of Year 1 versus Year 2 showed negligible variation with shift of only +0.8 degrees Celsius and minimal distances ($d_B = 0.4, W_2 = 0.9$), indicating a Highly Stable system also spatial comparison between Urban also Rural zones revealed shift of +14.5 degrees Celsius but significantly higher geometric divergence, with d_B of 12.1 and W_2 of 18.4, proving that the topological signatures are Distinct.

4.6. Differential Geometry of Thermal Flux also Helmholtz-Hodge Decomposition

We changed analysis from static scalar view to dynamic vector calculus system by considering Land Surface Temperature (LST) as scalar potential field (Φ). To understand how heat flows, which shows the direction of the fastest temperature increase also to focus on how heat builds up versus how it spreads out, we used the Laplacian Operator (nabla-squared), solving steady-state Poisson Equation for heat movement in urban solid medium:

$$\nabla^2 \Phi = \text{divergence}(\nabla \Phi) = \left(\frac{\partial^2 \Phi}{\partial x^2}\right) + \left(\frac{\partial^2 \Phi}{\partial y^2}\right) = -\left(\frac{q_{gen}}{k_{thermal}}\right) \quad (33)$$

The Laplacian serves as a measure of topological convexity in a high-dimensional space; our work maps divergence across an area with clearly defined source regions (signified by extremely negative Laplacian values, $\nabla^2 \Phi \ll 0$) displaying local convective peaks from thermal manifold curves. By plugging in the thermal conductivity value ($k=0.75 \text{ W/mK}$) from Karbala's asphalt mix and measuring recorded fluxes, we identified a key diagnostic point of divergence at the intersection of Al-Jumhuriyah street; this indicates evidence of an active exothermic heat generation process occurring there. Contrarily, sink regions were found with a positive Laplacian ($\nabla^2 \Phi > 0$) located in the deeper urban canyons formed by alleyways within the Old City.

$$F = -\nabla (\text{Scalar}_{potential}) + \text{curl}(\text{Vector}_{potential}) \quad (34)$$

The scalar potential deals with how heat moves outward from hot surfaces also the vector potential deals with the swirling, chaotic air movements that arise near rough surfaces where our data suggest that Solenoidal Energy Component has biggest impact where marble meets asphalt also this makes zone of thermal instability where the thermal field has a non-zero curl ($\text{curl } F \neq 0$) also this, in turn, generates small advective swirls that trap pollutants and heat where people walk.

4.7. Multifractal Spectral Analysis also Lacunarity of Isotherms

Euclidean geometry does not fully represent the rough nature of the thermal interface in Karbala also we used Multifractal Detrended Fluctuation Analysis (MF-DFA) to show how the thermal field scales. Instead of a monofractal, which a single dimension describes, the urban heat island of Karbala has a Multifractal Spectrum $f(\alpha)$. Alpha is the singularity strength or Holder exponent also partition function $Z_q(s)$ scales with the segment size s as:

$$Z_q(s) = \text{summation } |\text{Probability}_i(s)|^q \text{ is proportional to } s^\tau(q) \quad (35)$$

The breadth of the multifractal spectrum ($\Delta_\alpha = \alpha_{\max} - \alpha_{\min}$) in our study shows how varied thermal conditions are also urban core has a wide spectrum ($\Delta_\alpha = 0.84$), meaning it is highly changeable and includes both gradual heat variations and distinct, very hot areas also we also found the generalized fractal dimension (D_q) for key isotherms also there's a dimensional phase shift at 45 degrees Celsius. Using box-counting variables and green belt area had dimension of $D_0 = 1.15$. But when we did the same calculation for the urban core isotherms above 45 degrees Celsius, the dimension was $D_0 = 1.68$, which is hyper-fractal dimension also this hyper-fractality means that as we look closer, ratio of perimeter to area gets very large in hot zones also we measured lacunarity (Lambda) as well, which tells us about gaps or texture:

$$\Lambda(r) = \left(\frac{\sigma_{mass}^2}{\mu_{mass}^2}\right) \quad (36)$$

4.8. Phase-Space Thermodynamics also Hysteresis Loops

To see how the system changes over time, we switched from looking at spatial maps to Phase-Space Trajectory Analysis also we charted the system's state on a 2D phase plane. X-axis shows the Land Surface Temperature $T(t)$ also Y-axis shows how fast the temperature changes (dT/dt) also paths created make closed loops, called Hysteresis Loops. Area inside these loops (A_{hyst}) shows how much energy is lost and relates to the district's Thermal Admittance also formula for this hysteresis comes from the energy balance:

$$C_{eff} * \left(\frac{dT}{dt}\right) = K_{down} * (1 - albedo) - \epsilon * \sigma * T^4 - H(T) \quad (37)$$

By finding the area of the hysteresis loop of LST compared to Solar Irradiance (K_{down}), we measured the Thermal Inertia (I). When we put in the typical daily values for the New Residential areas, which are built with reinforced concrete, we got a Hysteresis Loop area:

$$Integral_{Area} = 450 \text{ Kelvin} - \text{Watts per square meter}.$$

This shape means system stores lot of energy and has large Phase Lag also hottest temperature happens about 3.5 hours after sun is highest in sky.

4.9. Homological Scaffolding of Anthropogenic Exergy Destruction

We improved the way we find man-made heat by using Exergy Analysis instead of just subtracting values also we treated the city surface as a system that wastes energy, not just something that gives off heat. To find the Anthropogenic Heat Flux density (Q_F), we solved the energy balance equation in reverse, treating it as a minor change in the area:

$$Q_F(x, y) = R_{net} - (G_{soil} + H_{sensible} + LE_{latent}) \quad (38)$$

Then, we used Topological Persistence analysis on the Q_F data also we found specific homology generators (H_0 features) that are very intense but don't last in space also these are point-source entropy generators, like neighborhood electrical diesel generators and industrial HVAC exhaust ports. These point sources tend to follow a Power Law distribution or Zipf Law in terms of their intensity:

$$Probability(Intensity > x) \text{ is proportional to } x^{(-\gamma)} \quad (39)$$

For the Karbala data, gamma is about 2.3 also this shows that just a few Mega-Emitters create most of the man-made heat also we found that these sources are clustered together, especially around the commercial ring of the shrines. From a thermodynamic view, these areas produce the most Entropy.

4.10. Linear Spectral Unmixing and Sub-Pixel Thermal Coupling

To better see the differences on the city surface, we used Linear Spectral Mixture Analysis (LSMA) also we saw each pixel's spectral signature (R_{λ}) as a simple combination of pure Endmembers (Asphalt, Concrete, and Vegetation) weighted by how much of each was there (f_i) also we solved the equation for the spectral inversion using the constrained least squares method:

$$R_{pixel} = \text{summation}(f_i * R_{endmember_i}) + \text{error} \quad (40)$$

Using the spectral reflectance values for Karbala materials in the Landsat bands, the mixing equation for a typical pixel in the Old City is:

$$R_{OldCity} = 0.62 * R_{Brick} + 0.28 * R_{Asphalt} + 0.10 * R_{Shadow} + 0.02 \quad (41)$$

This shows that 62% of the radiative signal comes from the yellow clay brick used locally. Also, we found the Thermal Coupling Index (TCI), which measures how heat is exchanged between these materials. Putting the fractional abundances into the thermal interaction model, we saw that Asphalt (f_{asph}) and Concrete walls (f_{conc}) create a Thermal Cavity amplification factor (Γ):

$$\Gamma = 1 + (f_{asph} * f_{conc}) / (f_{veg} + 0.01) = 1 + (0.28 * 0.62) / 0.05 = 4.47 \quad (42)$$

This Γ value of 4.47 means that having paved roads next to building walls in Karbala makes the sensible heat flux almost 4.5 times higher than on an open surface also this confirms that how materials are arranged is more important than the materials themselves.

4.11. Radiative Transfer and the Urban Canyon Sky View Factor

We studied how well the narrow alleyways (Sikkak) around the Holy Shrines trap radiation by using the Sky View Factor (ψ_{sky}) also the SVF shows how much of the sky can be seen from the ground, which controls how well Longwave Nighttime Cooling (L_{up}) works also we used the geometric integration for an infinite urban canyon with height H and width W . In the Al-Abbas market district, where the ratio H/W is about 2.5, the equation is:

$$\psi_{sky} = \cos(\arctan(H / (0.5 * W))) = \cos(\arctan(5.0)) = 0.196 \quad (43)$$

With a Sky View Factor of only 0.196, less than 20% of the cold night sky is available for cooling also we then found the Effective Emissivity (ϵ_{eff}) of these canyons. Because photons bounce between the canyon walls, the apparent emissivity increases, trapping heat. Using the material emissivity of brick ($\epsilon_{brick} = 0.93$) and the SVF, the equation is:

$$\begin{aligned} \epsilon_{eff} &= \epsilon_{brick} + (1 - \psi_{sky}) * (1 - \epsilon_{brick}) = 0.93 + (1 - 0.196) * 0.07 \\ &= 0.986 \quad (44) \end{aligned}$$

The effective emissivity goes up to 0.986, which is close to a perfect Black Body ($\epsilon = 1.0$) also this shows that the design of the Old City turns regular building materials into a very good radiation trap.

4.12. Lyapunov Stability Analysis and Chaos Theory

To see if the Karbala Urban Heat Island (UHI) is a stable system or a chaotic one, we calculated the Maximal Lyapunov Exponent (λ_{max}) from the thermal data also these measures how quickly trajectories in phase space move apart also the trajectory divergence is:

$$|\delta Z(t)| \text{ is_approximately } e^{(\lambda_{\max} * t)} * |\delta Z_0| \quad (45)$$

Using the daily temperature data for the Periphery Transition Zones, we got a positive exponent:

$$\lambda_{\text{Transition}} = +0.42 \text{ per day} \quad (46)$$

A positive lambda of +0.42 means Chaotic Behavior. In these zones, small changes can lead to big differences in temperature.

$$\lambda_{\text{Core}} = -0.18 \text{ per day} \quad (47)$$

4.13. Spatial Autocorrelation and Local Moran Index

Local Moran I (I_i) was used to measure how thermal anomalies cluster spatially, testing for local spatial similarities also the formula for the i-th pixel, using the normalized weight matrix (w_{ij}) for the 8-neighbor adjacency, is:

$$I_i = \left(\frac{(x_i - \text{mean}_x)}{\sigma^2} \right) * \text{summation}(w_{ij} * (x_j - \text{mean}_x)) \quad (48)$$

For the Bayn al-Haramayn plaza, using the local mean temperature deviation of +12 degrees Celsius above the rural mean and the neighbor deviations:

$$I_{\text{plaza}} = (12 / 3.2) * \text{summation}(1 * 11.5) = 41.3 \quad (49)$$

This high positive value of $I_i = 41.3$ confirms a High-High Cluster, suggesting a consistent thermal structure spanning 1.2 kilometers. In contrast, the internal courtyards of the shrines appeared as Low-High Outliers with negative I_i , showing cool areas of 38 degrees Celsius surrounded by extreme heat of 55 degrees Celsius.

4.14. Exergo-economic Cost of Thermal Degradation

The thermodynamic losses were then converted into an Exergy Cost Function (C_{thermo}), which shows the work potential lost because of the irreversible entropy generation of the UHI also the urban atmosphere was modeled as a Carnot Heat Engine operating in reverse also the Exergy Destruction rate ($X_{\text{dot}_{\text{dest}}}$) per unit area was found using the ambient temperature ($T_0 = 310 \text{ Kelvin}$) and the surface temperature (T_s):

$$X_{\text{dot}_{\text{dest}}} = Q_{\text{net}} * \left(1 - \left(\frac{T_0}{T_s} \right) \right) \quad (50)$$

Using the peak values for the asphalt road network, where Q_{net} is about 650 Watts per square meter and T_s is 331 Kelvin:

$$\begin{aligned} X_{\text{dot}_{\text{Roads}}} &= 650 * (1 - (310 / 331)) = 650 * (0.063) \\ &= 41.2 \text{ Watts per square meter} \quad (51) \end{aligned}$$

This means that 41.2 Watts per square meter of energy is continuously lost by the paved surfaces also when added up over the city road network area of 5.8 square kilometers, the total Thermodynamic Loss (L_{total}) is:

$$L_{\text{total}} = 41.2 * 5.8 * 10^6 = 239 \text{ Megawatts} \quad (52)$$

This 239 Megawatts is like a power plant of heat working against the city's cooling systems.

5. Conclusions

The heat map of Holy Karbala illustrates that an Urban Heat Island (UHI) is a deterministic feature as opposed to being simply a random occurrence or occurrence of chance. Using the Land Surface Temperature (LST) method for heat identification, the research project finds that there are 4215 points of heat leading to the creation of 18 stable clusters of heat, which ultimately come together to form what is referred to as the Giant Thermal Component (GTC) and a Persistence Entropy (PE) value of 2.41, validating the fact that this phenomenon is a heat storage structure.

A major outcome of the study is that the Holy Shrines represent a persistent area of the topological hole, creating what is known as a Thermal Void, which has an average temperature of approximately 38.2 degrees Celsius, surrounding it are multiple circular areas of extreme heat with temperatures exceeding 56.4 degrees Celsius, thus indicating that the courtyard type provides an effective barrier against heat exchange. Additionally, the study identified a significant barrier termed the Suffocation Threshold, (threshold being 42 degrees Celsius). At that temperature, significant Peak Levels are produced whereby heat and pollution are trapped in a closed thermal loop. Overall, this configuration is stable as evidenced by the high Bottleneck Distance of 3.1, which signifies that the basic shape of the underlying topology does not vary with daily temperature change.

Therefore, mounting an effective response to the thermal crisis facing Karbala must address the problem through the lens of Topological Urban Design. A successful remedy to the heat crisis must incorporate the creation of green corridors to break-up the clusters of heat, as well as the redesign of the Urban Canyon's geometry to facilitate cooling airflow, especially at the identified Suffocation Threshold of 42 degrees Celsius. Consequently, as the crisis is structurally based within the Topology of Karbala, so too must the structural alteration of that geometry be implemented.

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