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Differential Properties of the Conformal Curvature Tensor in Fifth-Recurrent Finsler Space: A Lie and Berwald Approach

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Abstract

This research investigates the Lie-derivative and Berwald covariant derivatives of the conformal curvature tensor in the generalized fifth recurrent Finsler space. The study focuses on the mathematical properties and interrelations of these tensors, exploring the behavior of the conformal curvature tensor under various differential operators, including the Lie derivatives and Berwald covariant derivatives. This paper builds upon the definition for the conformal curvature tensor under the Lie derivative in generaralized $\mathfrak{B}K-$fifth recurrent Finsler space. We study the relations between the mentioned curvature tensors and R^i_{jkh} by Lie-derivative. The Lie-derivative for the conformal curvature tensor C_{ijkh} and The fifth-order Berwald covariant derivatives are mutually commutative. In addition, we prove that the conformal curvature tensor C_{ijkh} behaves as a fifth recurrent under certain conditions. In conclusion, We demonstrate that applying the fifth-order Berwald covariant derivative to the Lie derivative of the curvature scalar R is vanishing.

Subject Classification: 53B40, 53C60, 83C20.
Keywords: Lie-derivative L_v, Conformal curvature tensor C_{ijkh}, Generalized $\mathfrak{B}K-$fifth recurrent Finsler space.

1 Introduction and Preliminaries

The fundamental tensor and curvature tensors play a central role in describing the intrinsic geometry of Finsler spaces and their behavior under various transformations and covariant derivatives. Among these tensors, the conformal curvature tensor is particularly important as it characterizes conformal properties of the space and generalizes the Weyl tensor from Riemannian geometry, providing insight into the invariance of geometric structures under conformal change.

The relationships between various curvature tensors in generalized Finsler spaces of different orders have been investigated by several authors. Abdallah [1] studied the relationship between two curvature tensors in Finsler spaces, while Abdallah and Hardan [2] clarified the relation between the tensors P^h_{ijk} and R^h_{ijk} under two third-order connections. Moreover, AL-Qashbari et al. [7] discussed extensions and developments of generalized $\mathfrak{B}K$-recurrent Finsler spaces. The fundamental concepts and recent developments of Finsler geometry, including curvature structures, were also presented in [3].

Furthermore, Ahsan and Ali [5] investigated several relationships between the W-curvature tensor and other curvature tensors, including the conformal, conharmonic, and concircular curvature tensors. Several important theorems concerning curvature tensors were also obtained in [9, 10, 14]. In addition, Opondo and Gouin [12, 13, 15, 16] studied the Lie derivative in recurrent and birecurrent spaces and examined some of its geometric properties. Various tensor identities were also established in $G\mathfrak{B}K\text{-}5RF_n$ using Lie derivatives in [8]. The present work is devoted to investigating the differential properties of the conformal curvature tensor in generalized fifth-order recurrent Finsler spaces by means of Lie and Berwald covariant derivatives.

Let us explore a generalized $\mathfrak{B}K-$fifth recurrent Finsler space satisfying the relations[6]:

$$\begin{aligned} &\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\, R^r_{ikh} = \frac{1}{g_{rj}}\Big[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m C_{ijkh} \\ &+\frac{1}{2}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(\, g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right) \\ &+\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\Big[\frac{R}{6}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\Big]\Big] \end{aligned} \tag{1}$$

$$\begin{aligned} &\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R^i_{jkh} = a_{sqlnm} R^i_{jkh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k\, g_{jh}\right) \\ &- c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl}\right) \\ &- e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}\right) - 2b_{qlnm} y^r \mathfrak{B}_r\left(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\right) \end{aligned} \tag{2}$$

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{jfkh} = a_{sqlnm} R_{jfkh} \tag{3}$$

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{jk} = a_{sqlnm} R_{jk} \tag{4}$$

if and only if

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (C^i_{jt} H^t_{kh}) - a_{sqlnm}(C^i_{jt} H^t_{kh}) = 0 \tag{5}$$

$$\begin{aligned} &b_{sqlnm}\left(g_{hf}g_{jk} - g_{kf}g_{jh}\right) - 2b_{qlnm} y^r \mathfrak{B}_r\left(g_{hf}C_{jks} - g_{kf}C_{jhs}\right) \\ &- c_{sqlnm}\left(g_{hf}C_{jkn} - g_{kf}C_{jhn}\right) - d_{sqlnm}\left(g_{hf}C_{jkl} - g_{kf}C_{jhl}\right) \\ &- e_{sqlnm}\left(g_{hf}C_{jkq} - g_{kf}C_{jhq}\right) + \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (C_{jft} H^t_{kh}) \\ &- a_{sqlnm}\left(C_{jft} H^t_{kh}\right) = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} &(n-1)[b_{sqlnm}\, g_{jk} - 2b_{qlnm}\, y^r \mathfrak{B}_r C_{jks} - c_{sqlnm} C_{jkn} - \, d_{sqlnm} C_{jkl} \\ &-e_{sqlnm} C_{jkq}] + \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m (C^u_{jt} H^t_{ku}) - \, a_{sqlnm}(C^u_{jt} H^t_{ku}) = 0, \end{aligned} \tag{7}$$

respectively.

A conformal curvature tensor C_{ijkh} (known as Weyl conformal curvature tensor) is described as [5]:

$$R_{ijkh} = C_{ijkh} + \frac{1}{2}\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right) + \frac{R}{6}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right). \tag{8}$$

The non-zero metric tensor g_{ij} and Kronecker delta δ^i_h satisfy the relations [11]

$$a)\ g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \tag{9}$$
$$b)\ \delta^i_i = n.$$

The associate curvature tensor R_{ijkh} of R^i_{jkh} and the curvature scalar R are given by

$$a)\ R_{ijkh} = g_{rj}\ R^r_{ikh} \qquad \text{and} \qquad b)\ R_{jk}g^{jk} = R. \tag{10}$$

A Finsler space in which the Berwald connection parameter G^i_{kh} does not depend on the directional coefficients y^i is known as an affinely connected space [4]. Therefore, the affinely connected space is defined by one of the equivalent conditions [18, 19]

$$a)\ \mathfrak{B}_k g_{ij} = 0 \qquad \text{and} \qquad b)\ \mathfrak{B}_k g^{ij} = 0. \tag{11}$$

Using Eq. [(11)a] in Eq. [(1)] and taking the Lie-derivative of the result equation, we have

$$\begin{aligned} L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\ R^r_{ikh}) &= \frac{1}{g_{rj}}\Big[L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m C_{ijkh}) \\ &+\frac{1}{2}L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\,(\ g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih})] \\ &+L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\ \frac{R}{6}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\ ]\right]\Big] \\ &-\frac{1}{g_{rj}}[(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\ R^r_{ikh})L_v g_{rj}]. \end{aligned} \tag{12}$$

The Lie - derivative of a general mixed tensor field T^i_{jkh} is given by [20]

$$\begin{aligned} L_v T^i_{jkh} &= v^m\mathfrak{B}_m T^i_{jkh} - T^m_{jkh}\mathfrak{B}_m v^i + T^i_{mkh}\mathfrak{B}_j v^m + T^i_{jmh}\mathfrak{B}_k v^m \\ &\quad + T^i_{jkm}\mathfrak{B}_h v^m + \dot{\partial}_m T^i_{jkh}\mathfrak{B}_r v^m y^r. \end{aligned} \tag{13}$$

The Lie- derivative of the metric tensors g_{ij} and g^{ij} are vanishing [8, 17]

$$a)\ L_v g_{ij} = 0. \qquad \text{and} \qquad b)\ L_v g^{ij} = 0. \tag{14}$$

The Berwald derivative of non-zero contravariant vector v^m, is vanishing

$$\mathfrak{B}_j v^m = 0. \tag{15}$$

The primary objective is to investigate the behavior of the conformal curvature tensor under various differential operators and to establish connections with other curvature tensors, particularly Cartan's third curvature tensor. This paper examines the conformal curvature tensor in the framework of $G\mathfrak{B}K - 5RF_n$, employing Lie-derivatives and Berwald covariant derivatives to derive new relations and results.

2 Main Results

Certain properties and relationships between the conformal curvature tensor and other tensors in $G\mathfrak{B}K-5RF_n$ are analyzed through the Lie derivative in the main findings.

Theorem 2.1 *In $G\mathfrak{B}K-5RF_n$, the Berwald covariant derivative of the fifth order for the conformal curvature tensor C_{ijkh} is given by*

$$\begin{aligned}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m C_{ijkh} &= \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[R_{ijkh}-\frac{1}{2}\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{jk}R_{ih}\right)\\ &-\frac{1}{6}(n-3)g_{ih}R_{jk}]. \qquad (16)\end{aligned}$$

Proof. Taking the Lie - derivative of Eq. (8) and using Eqs. [(10)b], (1.13), [(14)a,b] and (15), we get

$$\begin{aligned}L_vC_{ijkh} &= v^m[\mathfrak{B}_mR_{ijkh}-\frac{1}{2}\left(g_{ik}\mathfrak{B}_mR_{jh}+g_{jh}\mathfrak{B}_mR_{ik}-g_{ih}\mathfrak{B}_mR_{jk}-g_{jk}\mathfrak{B}_mR_{ih}\right)\\ &-\frac{1}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)g^{jk}\mathfrak{B}_mR_{jk}].\end{aligned}$$

Using Eqs. [(9)a,b] in above equation, we get

$$\begin{aligned}L_vC_{ijkh} &= v^m[\mathfrak{B}_mR_{ijkh}-\frac{1}{2}\left(g_{ik}\mathfrak{B}_mR_{jh}+g_{jh}\mathfrak{B}_mR_{ik}-g_{jk}\mathfrak{B}_mR_{ih}\right)\\ &-\frac{1}{6}(n-3)g_{ih}\mathfrak{B}_mR_{jk}].\end{aligned}$$

Using Eqs. (13), (15) and [(11)a] in above equation, we get

$$\mathfrak{B}_mC_{ijkh}=\mathfrak{B}_m[R_{ijkh}-\frac{1}{2}\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{jk}R_{ih}\right)-\frac{1}{6}(n-3)g_{ih}R_{jk}].$$

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s, we get (16).

Theorem 2.2 *In $G\mathfrak{B}K-5RF_n$, the Lie-derivative for the conformal curvature tensor C_{ijkh} and the Berwald covariant derivative of the fifth order are commutative if and only if Eq.(20) holds.*

Proof. Taking the Lie - derivative of Eq. (8), we conclude

$$\begin{aligned}L_vR_{ijkh} &= L_vC_{ijkh}+\frac{1}{2}L_v\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)\\ &+L_v\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right].\end{aligned}$$

Taking Berwald covariant derivative of fifth order for above equation with respect to x^m, x^n, x^l, x^q and x^s, we get

$$\begin{aligned}&\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vR_{ijkh})=\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vC_{ijkh})\\ &+\frac{1}{2}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mL_v\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)\\ &+\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mL_v\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right].\end{aligned}$$

Using Eq. [(10)a] in above equation, we get

$$\begin{aligned}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vC_{ijkh}) &= \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_v(g_{rj}R^r_{ikh}))\\ &-\frac{1}{2}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mL_v\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)\\ &-\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mL_v\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right].\end{aligned} \tag{17}$$

Using Eqs. [(11)a] and [(14)a] in Eq. (12), we get

$$\begin{aligned}L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mC_{ijkh}) &= L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(g_{rj}R^r_{ikh})]\\ &-\frac{1}{2}L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)]\\ &-L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)]\right].\end{aligned} \tag{18}$$

In view of Eqs. (17) and (18), we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vC_{ijkh}) = L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mC_{ijkh}) \tag{19}$$

if and only if

$$\begin{aligned}&\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_v(g_{rj}R^r_{ikh}))\\ &-\frac{1}{2}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mL_v\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)\\ &-\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mL_v\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right] = L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(g_{rj}R^r_{ikh})]\\ &-\frac{1}{2}L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)]\\ &-L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)]\right].\end{aligned} \tag{20}$$

Therefore, equation (19) refers to required.

Theorem 2.3 *In $G\mathfrak{B}K-5RF_n$, the Lie-derivative is distributive on the addition of Berwald covariant derivative of the fifth order for R^r_{ikh} and C_{ijkh} is given by*

$$\begin{aligned}L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(R^r_{ikh}+C_{ijkh})] &= L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mR^r_{ikh})\\ &+L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mC_{ijkh})\end{aligned} \tag{21}$$

if and only if Eq. (24) *holds.*

Proof. Adding Cartan's third curvature tensor R^r_{ikh} of Eq. (8), we obtain

$$\begin{aligned}R^r_{ikh}+R_{ijkh} &= R^r_{ikh}+C_{ijkh}+\frac{1}{2}\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)\\ &+\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right).\end{aligned}$$

Taking Berwald covariant derivative of fifth order for above equation with respect to x^m, x^n, x^l, x^q and x^s, we get

$$\begin{aligned}&\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(R^r_{ikh}+R_{ijkh})=\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(R^r_{ikh}+C_{ijkh})\\&+\frac{1}{2}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)\\&+\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)].\end{aligned}$$

Taking the Lie-derivative of above equation, we get

$$\begin{aligned}&L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(R^r_{ikh}+C_{ijkh})]=L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(R^r_{ikh}+R_{ijkh})]\\&-\frac{1}{2}L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)]\\&-L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)]\right].\end{aligned}\tag{22}$$

Now, adding Eq. (12) with Eq. (18) and using Eq. [(14)a], we get

$$\begin{aligned}&L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mR^r_{ikh})+L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mC_{ijkh})\\&\qquad=\frac{1}{g_{rj}}L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mC_{ijkh})\\&+\frac{1}{2}(\frac{1}{g_{rj}}-1)L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)]\\&+(\frac{1}{g_{rj}}-1)L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)]\right]\\&+L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(g_{rj}R^r_{ikh})].\end{aligned}\tag{23}$$

In view of Eqs. (22) and (23), we get (23) if and only if

$$\begin{aligned}&L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(R^r_{ikh}+R_{ijkh})]=\frac{1}{g_{rj}}L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mC_{ijkh})\\&+\frac{1}{2}(\frac{1}{g_{rj}})L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)]\\&+(\frac{1}{g_{rj}})L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)]\right]+L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(g_{rj}R^r_{ikh})].\end{aligned}\tag{24}$$

This proof is complete.

Theorem 2.4 *In $G\mathfrak{B}K-5RF_n$, the conformal curvature tensor C_{ijkh} behaves as fifth recurrent [provided Eqs.* (27), (6) *and* (7) *hold].*

Proof. Transvecting Eq.(8) by the non-zero tensor $g^{rk}g^{mh}$ and using Eq. [(9)a], we get

$$C_{ijkh}=R_{ijkh}.\tag{25}$$

Using Eqs.(3), (4) and [(11)a] in Eq. (16), we get

$$\begin{aligned}\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_mC_{ijkh}\quad&=\quad a_{sqlnm}[R_{ijkh}-\frac{1}{2}\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{jk}R_{ih}\right)\\&\qquad-\frac{1}{6}(n-3)g_{ih}R_{jk}].\end{aligned}$$

Or

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m C_{ijkh} = a_{sqlnm} R_{ijkh} \tag{26}$$

if and only if

$$a_{sqlnm}\left[\frac{1}{6}(n-3)g_{ih}R_{jk} + \frac{1}{2}\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{jk}R_{ih}\right)\right] = 0. \tag{27}$$

Using Eq. (25) in Eq. (26), we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m C_{ijkh} = a_{sqlnm} C_{ijkh}, \tag{28}$$

Therefore, we get the required.

Now we focus on important corollaries derived from previous theories. In view of Eq. (16), we have

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m C_{ijkh} = \mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m R_{ijkh} \tag{29}$$

if and only if

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left[\frac{1}{6}(n-3)g_{ih}R_{jk} + \frac{1}{2}\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{jk}R_{ih}\right)\right] = 0. \tag{30}$$

Thus, we get

Corollary 2.5 *In $G\mathfrak{B}K-5RF_n$, the Berwald covariant derivative of the fifth order for the conformal curvature tensor C_{ijkh} and the associate curvatur tensor R_{ijkh} are equal if and only if Eq.* (30) *holds.*

Using Eqs. [(11)a] and [(14)a] in Eq. (18), we get

$$L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m C_{ijkh}) = g_{rj} L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(R^r_{ikh})], \tag{31}$$

if and only if

$$\begin{aligned}&\frac{1}{2}L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right)] \\ &+L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\frac{R}{6}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)]\right] = 0.\end{aligned} \tag{32}$$

Therefore, we conclude

Corollary 2.6 *In $G\mathfrak{B}K-5RF_n$, the Lie-derivatives of Berwald covariant derivative of the fifth order for the conformal curvature tensor C_{ijkh} and R^r_{ikh} are co-directional [provided Eq.* (32) *holds].*

Using Eqs. (2),[(9)a] and [(14)a] in Eq (12), we get

$$\begin{aligned}L_v(a_{sqlnm}R^r_{ikh}) = \frac{1}{g_{rj}}\Big[&L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m C_{ijkh}) \\ &+\frac{1}{2}L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\left(\, g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right)] \\ &+L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\frac{R}{6}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)]\right]\Big].\end{aligned}$$

Or

$$L_v(R^r_{ikh}) = \frac{1}{(a_{sqlnm}\ g_{rj})}\Big[L_v(\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m C_{ijkh}) \tag{33}$$
$$+\frac{1}{2}L_v[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m\,(\ g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih})]$$
$$+L_v\left[\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m[\ \frac{R}{6}\,(g_{ih}g_{jk} - g_{ik}g_{jh})\ ]\right]\Big] - \frac{1}{a_{sqlnm}}R^r_{ikh}L_v(a_{sqlnm}),$$

if and only if Eq (3) holds. Thus, we conclude

Corollary 2.7 *In $G\mathfrak{B}K - 5RF_n$, the Lie-derivative for R^r_{ikh} is given by Eq (33) [provided Eq (3) holds].*

Transvecting Eq.(8) by the tensor g^{jm}, and using Eqs. [(9) a] and (25), we get

$$g_{ik}R_{jh}g^{jm} - g_{ih}R_{jk}g^{jm} = 0.$$

Contracting the indices m and h in above equation and using Eqs. [(9) a] and [(10)b], we get

$$g_{ik}R = 0.$$

Now, taking the Lie-derivative of above equation and using [(14)a], then taking Berwald covariant derivative of fifth order for result equation with respect to x^m, x^n, x^l, x^q and x^s and using Eq. [(11)a], we get

$$\mathfrak{B}_s\mathfrak{B}_q\mathfrak{B}_l\mathfrak{B}_n\mathfrak{B}_m(L_vR) = 0. \tag{34}$$

Thus, we infer

Corollary 2.8 *In $G\mathfrak{B}K - 5RF_n$, the Berwald covariant derivative of the fifth order of the Lie derivative for the curvature scalar R is vanishing.*

3 Conclusion

This study investigated the relationships between Cartan's third curvature tensor R^i_{jkh} and other curvature tensors, especially conformal curvature tensor, by the Lie derivative in $G\mathfrak{B}K - 5RF_n$. Furthermore, we proved that the conformal curvature tensor C_{ijkh} behaves as a fifth recurrent under certain conditions in the main space by using the Lie derivative. In future, We expand this work by using the Lie-derivative in generalized n^{th} recurrent Finsler space in Berwald and Cartan senses.

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