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ON BOUNDEDNESS AND GLOBAL ASYMPTOTIC STABILITY OF SOLUTIONS OF DUFFING EQUATION

By:

Ezugorie Ikechukwu Godwin

Department of Mathematics, Enugu State University of Science and Technology

Email: ikegodezugorie@esut.edu.ng

Abstract

This paper investigated boundedness and global asymptotic stability of solutions of a class of certain second order nonlinear differential equation with damping using the Lyapunov second order and eigenvalue approach. Through the exploits of Schwartz inequality and assumptions on the inhomogeneous part, solutions of the Duffing equation were bounded and the equilibrium point was global asymptotically stable. Response to damping revealed that the damping effect was not negligible thereby reducing oscillations. Application of our results can be seen in the construction of door net where the trajectory returns to equilibrium as fast as possible. Furthermore, Mathcad software was used to analyze the behavior of the system which extends some results in literature.

Keywords:

Boundedness, Duffing equation, Global asymptotic stability.

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1. Introduction

Duffing equation is a second order differential equation that is widely used in physics, economics, engineering, and many other phenomena. The study is significant because of the physical application of the results. It is significant to the Physicist who uses it to study propagation of wave in mobile phones, radios, televisions [1]. The signal processor will find its significance in modeling of the nonlinear spring mass system [2], modeling of the ultra wide band (UWB) radio systems for detecting high speed wireless [3], fussy modeling and the adaptive control of uncertain chaotic system.

It is also significant to the Engineers who find its applicability in energy harvesting [4], crash analysis [5], modeling conservative double well oscillator which occur in magneto-elastic mechanical system [6] and prediction of emission characteristics of sawdust particle [18]. It is also significant to the medical and life scientist who will find it applicable in the modeling of the brain [8] and prediction of the heart beats (pulse). The environmentalist will find it applicable in predicting earthquake occurrences [9] and other natural disasters such as tsunamis and heat waves. It is used to model plant systems, where the effect of nonlinear stiffness on resonant behavior of plants is described by the Duffing oscillators with hardening non-linearization. Its important can be seen in signal processing [10] and prediction of weather condition. In Biology, [11] opined that common feature of oscillating biological systems are feedback loops, negative feedback and genetic oscillation. [12] found that the low-frequency property of a capillary oscillation play a vital role in mass and energy transmission in blood flow, permeability and cell growth. The low frequency is also widely used in energy harvesting device [13, 14].

Stability is one of the qualitative properties of a differential equations that is an important factor in nonlinear analysis. For nonlinear differential equation of the Duffing type, the stability results are obtained by observing the behavior of the trajectory around the equilibrium point. For instance, see [15, 16, 17] and their references therein. For stable equilibrium point, the behavior of the trajectory is predicted around the equilibrium point after a little displacement from the origin. For asymptotic stable equilibrium point, the behavior of the trajectory is predicted close to the equilibrium point while for global asymptotic stable equilibrium point, the behavior of the trajectory is predicted as a little displacement from the origin to infinity. However, these results are necessary for explaining the importance of stability of Duffing equation such oscillation of rigid pendulum using moderately large amplitude motion [18], vibration of buckled beam [19], choice of hard/soft spring in the mechanism of shock absorbers [20] and the inherent pull-in instability micro-electromechanical systems (MEMS)s which can be overcome by the fractal vibration theory [21, 22].

In this paper, we consider the Duffing equation of the form

$$x'' + cx' + g(t\,,x) = \lambda p(t) \qquad (1)$$

where $g(t\,,x) = \lambda x + \beta x^3$, c is the damping coefficient, $g : \mathbb{R} \longrightarrow \mathbb{R}$ is sufficiently smooth, $g(-\infty) = \lim_{x\to-\infty} g(x)$ and $g(+\infty) = \lim_{x\to\infty} g(x)$ satisfies the inequality

$$g(-\infty) < g(x) < g(+\infty) \quad \text{for all} \quad x \in \mathbb{R} \qquad (2)$$

The function $p \in L^{\infty}(\mathbb{R})$ is bounded, aperiodic and has a well-defined mean value $\bar{p} \in \mathbb{R}$

defined by

$$\bar{p} = \lim_{r\to+\infty} \left(\frac{1}{\tau}\right) \int_a^{a+\tau} p(t)\,dt \quad (\text{uniformly in } \; a \in \mathbb{R}) \qquad (3)$$

In [23],

$\bar{p} = \lim_{r\to+\infty} \left(\frac{1}{T}\right) \int_a^{T} p(t)\,dt$ was used to prove that Landesman-Lazer type condition

$$g(-\infty) < g(x) < g(+\infty) \qquad (4)$$

Equation (4) was necessary and sufficient for existence and boundedness of the solution in

$x \in W^{2,\infty}(\mathbb{R})$. Every solution of equation (1) is bounded in $x \in W^{2,\infty}(t_0\,,\,\infty)$ when equation (3) is satisfied. In our case, we observed that the existence of total derivative of the Lyapunov function is necessary and sufficient to confirm boundedness of the equation (1) in $W^{2,2}(\mathbb{R})$. Related results have been previously obtained by [24, 25, 26, 27, 28, 29]. It is also worthy to observe that the Landesman-Lazer condition are necessary but not sufficient to guarantee global asymptotic stability. For instance, if g holds for equation (2) and nonincreasing for some p satisfying equation (4), then equation (1) is not globally asymptotically stable. Related problem have been studied by [30, 31] with resounding results. For instance, [30, 31] achieved local stability by assuming that p is periodic of a fixed period. In our own case, Lyapunov boundedness theorem will be used to overcome all these obstacles and achieve global asymptotic stability. To achieve global asymptotic stability, [32] considered the equation

$$x'' + cx' + g(x) = p(t) \qquad (5)$$

where $p \in L^1(\mathbb{R})$ and the parameters are not restricted in such case. In this work, we assumed that $p \in L^2(\mathbb{R})$ and restrictions are imposed on the damping coefficient to investigate damping effect on the global asymptotic stability of equation (1). In [33], global asymptotic stability of a second order differential equation was achieved under the assumptions placed on the damping coefficient and the nonlinear part. Some methods have been employed by few researchers to study boundedness and global asymptotic stability of nonlinear differential equation. Control theory approach was used by [34] to establish boundedness of the equation.

$$x'' + cx' + \alpha(t)y = 0 \qquad (6)$$

Where

$\alpha \in L^2(\mathbb{R}^+) \quad a \leq \alpha(t) \leq b \qquad a.et \in \mathbb{R}^+$

While comparison technique was employed by [23] to show that the solution of the differential equation of the form (1) can be compared directly once there is restriction on a finite interval. Control theory approach is limited because it places so much importance on the interval relative to the parameters. The limitation of the comparison technique is that it is impossible to compare different or completely identical qualitative properties. However, Lyapunov second method and its theorem will be employed in this paper to study boundedness and global asymptotic stability of solutions of Duffing equation. Lyapunov second method has advantage over these methods because it is general and appropriate for dealing with uncertain systems and nonlinear limit-varying parameters.

The objective of this paper therefore, is to investigate boundedness and global asymptotic stability of solutions of Duffing equation. We further analyze different type of damping for the Duffing equation and examine the effects of damping on the system.

Motivation of this work are the works in [20] and [35]. This article is organized in this format: the next section gives the basic preliminary surrounding the subject while section three is dedicated to known results in the theory. The numerical simulations features in section four and the last section is conclusion.

2. Preliminary Results

Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous in a given domain.

Definition 2.1 *Bounded Function*

f is said to be bounded on the domain of definition $D(f)$ if there exist $M > 0 : |f(x)| \leq M$ for all x element of the domain of f.

In other words, f is bounded if $\exists\, m$ and $M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in D(f)$. These numbers m and M are respectively lower and upper bounds.

Definition 2.2 *Lyapunov Stability*

Consider the system of first order differential equation

$$\dot{X} = F(t, X), \qquad X(t_0) = X_0 \qquad (7)$$

where $X(t) \in D(F) \subseteq \mathbb{R}^\alpha$ is the system state vector, D is an open set with origin, and $f : D \longrightarrow \mathbb{R}^\alpha$ is continuous in D. Assuming F has an equilibrium at X_e so that $\mathrm{F}(x) = 0$, then system (6) is said to be stable if for any $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|\mathrm{X}(0) - X_e\| < \varepsilon, \quad \forall\ t \leq 0\,.$$

If δ is not depending on t_0 over entire time interval i. e. $\delta = \delta(\varepsilon)$ then the solution of (1) is said to be uniformly stable.

Definition 2.3 *Asymptotic Stability*

Solution of (1) *is asymptotically stable if the system is Lyapunov stable and* $\delta = \delta(t_0) > 0$ *and* $\lim_{t \to \infty} \|X(t) - X_e\| = 0$

Definition 2.4 *Global Asymptotic Stability*

Solution of equation (1) *is globally asymptotically stable, if there exists* $\delta = \delta(t_0) > 0$ *such that*

$$\|X(0) - X_e\| < \delta \implies \|X(t) - X_e\| = 0 \quad as \quad t \to \infty$$

Furthermore, if the convergence in definition (2.4) does not depend on the initial state $X(t_0)$

over the entire special domain, then the solution of the system is globally asymptotically stable

Definition 2.5 *Let* $V : \mathbb{R}^\alpha \longrightarrow \mathbb{R}$ *be continuously differentiable with* $V(0) = 0$. *Then* V *is*

1. *Positive definite if* $V(X) > 0$ $\forall X \neq 0$ *and* $V(0) = 0$
2. *Negative definite if* $V(X) < 0$ $\forall X \neq 0$ *and* $V(0) = 0$
3. *Positive Semi-definite if* $V(X) \geq 0$ *and vanish for some* $X \neq 0$
4. *Negative Semi-definite if* $V(X) \leq 0$ *and vanish for some* $X \neq 0$

Definition 2.6 *Lyapunov Function*

A scalar function $V : \mathbb{R}^\alpha \longrightarrow \mathbb{R}$ *is called Lyapunov function if it is positive definite (Semi-definite) and its derivative with respect to trajectories of a given system is negative definite (Semi-definite)*

Definition 2.7 *Damping*

This is the resistance offered to the oscillation.

Theorem 2.1 *Consider the differential equation*

$$\dot{X} = F(t, X), \qquad X(t_0) = X_0 \tag{8}$$

where $F : \mathbb{R}^\alpha \longrightarrow \mathbb{R}^\alpha$ *and* $\dot{X} = \frac{dX}{dt}$, *If* $V : \mathbb{R}^\alpha \longrightarrow \mathbb{R}$ *is a continuously differentiable and satisfies the following conditions*

1. $V(X)$ *is positive definite.*
2. *The time derivative of* V *(that is* $\dot{V}$*), is negative semi-definite*

Then the equilibrium point is stable in the sense of Lyapunov.

Theorem 2.2 *Let* $V : \mathbb{R}^\alpha \longrightarrow \mathbb{R}$ *is a continuously differentiable and satisfies the following conditions*

1. $V(X)$ *is positive definite.*
2. *The time derivative of* V *(that is* $\dot{V}$*), is negative semi-definite*

Then the equilibrium point is asymptotically stable in the sense of Lyapunov.

Theorem 2.3 *Let* V *be a Lyapunov function, which satisfies the following*

1. *All sub-level sets of* V *are bounded*
2. $\dot{V} \leq 0 \quad \forall X$

Then $\exists K$ *such that* $\|X(t)\| \leq K, \quad \forall t$

Theorem 2.4 *Suppose that there is* V *for* $X = 0$ *and* V *is radically unbounded unbounded that is* $\|x\| \to \infty \implies V(x) \to \infty$ *, then the equilibrium point is globally asymptotically stable.*

3. Main Results

In this section, we present the main tool for establishing global asymptotic stability and boundedness of solutions of Duffing equation.

3.1 Stability of Solutions

Consider the general Duffing equation

$$\ddot{x} + c\dot{x} + g(t, x) = \lambda p(t) \qquad (9)$$

when $g(t, x) = \lambda x + \beta x^3$

equation (9) becomes

$$\ddot{x} + c\dot{x} + \lambda x + \beta x^3 = \lambda p(t) \qquad (10)$$

Our main results are given as below:

Theorem 3.1 *Let* β *and* λ *be positive constants, then all solutions of the corresponding homogeneous equation to equation* (10) *for* $p(t) = 0$ *is asymptotically stable.*

Theorem 3.2 *Subject to the conditions of theorem* (3.1) *, the solution of equation* (10) *are bounded and in fact uniformly bounded.*

Theorem 3.3 *Let* $g = \infty$ *then solutions of equation* (10) *is globally asymptotically stable.*

Theorem 3.4 *In addition to the conditions of theorem* (3.1) *, suppose that* $p(t, x, \dot{x}) = p(t)$ *and*
$|p(t)| \leq M$ *, for all* $t \leq 0$ *, then there exists a constant* σ *,* $(0 < \sigma < \infty)$ *depending only on the constants* β *and* λ *such that every solution of* (10) *satisfies*

$$x^2(t) + \dot{x}(t) \leq (e^{-\sigma t} (A_1 + A_2 \int_{t_0}^{t} |p(\tau)| e^{1/2\sigma t} d\tau)^2) \qquad (11)$$

for all $t \geq t_0$, where the constant $A_1 > 0$ depends on β *and* λ as well as on t_0 , $x(t_0)$, $\dot{x}(t_0)$; and the constant $A_2 > 0$ depends on β *and* λ *.*

Equation (10) can be put in its equivalent system as $\dot{x} = y$

$$\dot{y} = -cy - \lambda x - \beta x^3 + \lambda p(t) \qquad (12)$$

To prove the theorems, we make use of the scalar function $V = V(x, y)$ defined as

$$2V = \frac{\delta}{\alpha c}\{[c^2 + c\lambda(\alpha + 1) + \alpha a\beta]x^2 + 2cxy + (\alpha + 1)y^2\} \qquad (13)$$

where a , c , α , β and δ are all positive with $|x^2| \leq a$ and $\alpha = \lambda + a\beta$. To prove the theorem, we need to establish that the function V is indeed a Lyapunov function.

Clearly, $V(0, 0) = 0$. Rearranging equation (13) we have

$$2V = \frac{\sigma}{\alpha c}([c^2 + \alpha(\lambda c + a\beta)]x^2 + c\lambda x^2 + 2cxy + \alpha y^2) \qquad (14)$$

On further simplification, we have

$$2V = \frac{\sigma}{\alpha c}([c^2 + \alpha(\lambda c + a\beta)]x^2 + c\lambda x^2 + cy(x + y)^2 + (\alpha + 1 - \frac{c}{\lambda})y^2)$$

$$2V = \frac{\sigma}{\alpha c}([c^2 + \alpha(\lambda c + a\beta)]x^2 + c\lambda x^2 + cy(x + y)^2 + (\frac{\lambda(\alpha + 1) - c}{\lambda})y^2)$$

Clearly,

$2V \geq \frac{\sigma}{\alpha c}([\,c^2 + \alpha(\lambda c + \alpha\beta)]x^2 + (\frac{\lambda(\alpha+1)-c}{\lambda})y^2)$ (15)

and using the Schwartz inequality

$|xy| \leq \frac{1}{2}(x^2 + y^2)$, we have that

$2V \leq \frac{\delta}{\alpha c}([c^2 + c\lambda(\alpha + 1) + \alpha\alpha\beta + c]x^2 + (\alpha + 1 + c)y^2)$ (16)

from equation (15) and (16) we have for K_1 and K_2

$K_1(x^2 + y^2) \leq V(x\,,\ y) \leq K_2(x^2 + y^2)$ (17)

Where

$$K_1 = \frac{\sigma}{\alpha c} \text{ x } \{\,[\,c^2 + \alpha(c\lambda + \alpha\beta)]\,,\left(\frac{\lambda(\alpha+1)-c}{\lambda}\right)\}$$

and

$$K_2 = \frac{\sigma}{\alpha c} \times \ max\ \{\,[\,c^2 + c\lambda(\alpha + 1) + \alpha\alpha\beta + c\,]\,,\left(\frac{\lambda(\alpha+1)-c}{\lambda}\right)\}$$

Inequality (17) shows the positivity of the function V. Differentiating (13) along the solution trajectories of system (12) , we have

$\dot{V} = \frac{\sigma}{\alpha c}(\,[\,c^2 + c\lambda(\alpha + 1) + \alpha\alpha\beta]x\dot{x} + c(x\dot{y} + \dot{x}y) + (\alpha + 1)y\dot{y})$ (18)

$\dot{V} = \frac{\sigma}{\alpha c}(\,[\,c^2 + c\lambda(\alpha + 1) + \alpha\alpha\beta]xy + cy^2 + [\,cx + (\alpha + 1)y]\,(-cy - \lambda x - \beta x^3 + \lambda p(t\,)))$ (19)

Simplifying using the conditions on the theorem, we have that when $p(t\,) \equiv 0$

$\dot{V} = -\delta(x^2 + y^2) \leq 0$ (20)

Inequality (20) shows that the time derivative of V is negative semi-definite.

When $p(t\,) \neq 0$ we have the time derivative of the scalar function as

$\dot{V} = -\delta(x^2 + y^2) + K_3(x^2 + y^2)^{1/2}|p(t)|$ (21)

where $K_3 = \sqrt{2} \times \lambda \times \min(c\,,\ (\alpha + 1)\,)$.

Proof of theorem (3. 1)

Proof:

From the Lyapunov stability theorem, the condition for V are satisfied by the Lyapunov candidate in equation (13) , we have

$$(x^2 + y^2) \leq \frac{V(x\,,y)}{K_1} \quad (22)$$

Using the above inequality in (21) we have that

$$\dot{V} \leq -\frac{\delta}{K_1}\left(V - K_3V^{1/2}\right)|p(t)|) \leq 0 \quad (23)$$

Equation (23) can simply be put as

$$\dot{V} \leq K_4V^{1/2}\left(|p(t)| - \frac{V^{1/2}}{K_3}\right)$$

where $K_4 = K_3 \times \frac{\delta}{K_1}$ when $p(t) \leq \frac{V^{1/2}}{K_3}$ we have

$$\dot{V} \leq 0 \quad (24)$$

Hence, the conclusion of theorem (3.1)

Proof of theorem (3. 2):

Now we consider

$$V = \frac{\sigma}{2\alpha c}(\,[\,c^2 + \alpha(\lambda c + \alpha\beta)]x^2 + (\frac{\lambda(\alpha+1)-c}{\lambda})y^2)$$

$$V = \frac{\sigma}{2\alpha c}(\,c^2 + \alpha(\lambda c + \alpha\beta)x^2 + \int_0^y kg(x)\,dx \quad (25)$$

where

$$K = \frac{\sigma}{2\alpha c}\left(\frac{\lambda(\alpha+1)-c}{\lambda}\right)$$

and

$$g(x) = 2x$$

Assume g satisfies $\int_0^y kg(x)\,dx \to \infty$ as $\|x\| \to \infty$ then equation (25) is radically unbounded and $\dot{V} \le 0$ in $\mathbb{R}^{\alpha}$. Hence, the solution is globally asymptotic stable.

Proof

However, if $p(t) \ge \frac{V^{1/2}}{K_3}$, by rewriting inequality (23)

$$\dot{V} \le -2K_5V + K_6V^{1/2}\,|p|$$

we have

$$\dot{V} + K_5V \le -K_5V + K_6V^{1/2}\,|p|$$
$$\dot{V} + K_5V \le K_6V^{1/2}\,(|p| - K_7V^{1/2})$$

where $K_5 = \frac{\delta}{2K_1}$, $K_6 = \frac{K_3}{K_1}$ and $K_7 = \frac{K_5}{K_6}$

Now we are considering the case when $p(t) \ge K_7V^{1/2}$, we have

$$\dot{V} + K_5V \le K_6V^{1/2}\,|p|$$

which implies

$$V^{-1/2}\dot{V} + K_5V^{1/2} \le K_6|p| \qquad (26)$$

Multiplying both sides of the inequality (26) by $e^{\frac{1}{2}K_5t}$ gives

$$e^{\frac{1}{2}K_5t}(\,V^{-1/2}\dot{V} + K_5V^{1/2}\,) \le e^{\frac{1}{2}K_5t}|p| \qquad (27)$$

That is

$$2\frac{d}{dt}(\,V^{\frac{1}{2}}\,e^{\frac{1}{2}K_5t}) \le e^{\frac{1}{2}K_5t}|p| \qquad (28)$$

Integrating both sides of (28) from t_0 to t gives

$$[\,V^{\frac{1}{2}}\,e^{\frac{1}{2}K_5t}\,]_{t_0}^{t} \le \int_{t_0}^{t} e^{\frac{1}{2}K_5t}|p(\tau)d\tau| \qquad (29)$$

which implies that

$$V^{\frac{1}{2}}(t)e^{\frac{1}{2}K_5t} \le V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_5t_0} + \frac{1}{2}K_6\int_{t_0}^{t}|p(\tau)|e^{\frac{1}{2}K_5\tau}d\tau \qquad (30)$$

or

$$V^{\frac{1}{2}}(t) \le e^{-\frac{1}{2}K_5t}\,\{\,V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_5t_0} + \frac{1}{2}K_6\int_{t_0}^{t}|p(\tau)|e^{\frac{1}{2}K_5\tau}d\tau\,\} \qquad (31)$$

On utilizing inequalities (15) and (17) we have

$$K_1(\,x^2(t) + \dot{x}^2(t)) \le e^{-K_5t}(\,(K_2(\,x^2(t_0) + \dot{x}^2(t_0)\,)\,)^{\frac{1}{2}}e^{\frac{1}{2}K_5t_0} + \frac{1}{2}K_6\int_{t_0}^{t}|p(\tau)|e^{\frac{1}{2}K_5\tau}d\tau)^2 \qquad (32)$$

for all $t \ge t_0$, thus

$$(\,x^2(t) + \dot{x}^2(t)) \le \frac{1}{K_1}\,\{\,e^{-K_5t}(\,(K_2(\,x^2(t_0) + \dot{x}^2(t_0)\,)\,)^{\frac{1}{2}}e^{\frac{1}{2}K_5t_0} + \frac{1}{2}K_6\int_{t_0}^{t}|p(\tau)|e^{\frac{1}{2}K_5\tau}d\tau)^2\,\}$$

$$\le e^{-K_5t}(\,A_1 + A_2\int_{t_0}^{t}|p(\tau)|e^{\frac{1}{2}K_5\tau}d\tau)^2 \qquad (33)$$

where A_1 and A_2 are constants depending on K_1, K_2, $K_3, \ldots$, K_7 and $(\,x^2(t) + \dot{x}^2(t))$.

By substituting $K_5 = \sigma$ in the inequality (33) we have

$$x^2(t) + \dot{x}^2(t) \le e^{-\sigma t}(\,A_1 + A_2\int_{t_0}^{t}|p(\tau)|e^{\frac{1}{2}\sigma\tau}d\tau)^2 \qquad (34)$$

This shows that the solutions of the equation are bounded

3.2 Response to damping

The auxiliary equation of equation (10) when $p(t) = 0$ is given by

$$s^2 + cs + \mu = 0 \qquad (35)$$

Where s represents the root of the equation and The characteristic root of equation (35) is given by

$$s = \frac{-c \pm \sqrt{c^2 - 4\mu}}{2} \qquad (36)$$

For different form of damping, the following cases with respect to the sign of the discriminate are considered.

Case 1: If $c^2 < 4\mu$ then $c^2 - 4\mu < 0$ shows that that the damping constant is small relative to μ. The term under the square root is negative and characteristic root become a complex root with general solution

$$s = C_1 e^{\frac{-ct}{2}} \cos(rt) + C_2 e^{\frac{-ct}{2}} \sin(rt) \qquad (37)$$

For arbitrary C_1, C_2 and $s = \frac{\sqrt{c^2-4\mu}}{2}$ is called the damped angular frequency of the system. $\cos(rt)$ shows oscillation and $e^{\frac{-ct}{2}}$ with negative exponent gives the decaying amplitude. As $t \to \infty$, the exponential goes asymptotically to zero. When $c = 0$, the response is a sinusoid. When the damping constant is small, the system is expected to still oscillate but with nonincreasing amplitude as its energy is converted to heat. Over time, the system should come to rest at at equilibrium. Since the root have nonzero imaginary part, the system is under-damped.

Case 2: If $c^2 > 4\mu$ then $c^2 - 4\mu > 0$ show that c is large relative to μ. $c^2 - 4\mu$ is positive and

$$s_1 = \frac{-c + \sqrt{c^2-4\mu}}{2} \qquad (38)$$

$$s_2 = \frac{-c - \sqrt{c^2-4\mu}}{2} \qquad (39)$$

with general solution

$$s = C_3 e^{s_1} + C_4 e^{s_1} \qquad (40)$$

for arbitrary C_3 and C_4. When the damping constant is large, frictional force is so great that the system cannot oscillate. This is a typical behavior of an unforced overdamped harmonic oscillator which does not oscillate. Since the characteristic root are real and distinct, the system is overdamped.

Case 3: If $c^2 - 4\mu$ then $c^2 - 4\mu = 0$ shows that the damping constant is between the overdamped and underdamped. The characteristic polynomial has a repeated root with general solution

$$e^{\frac{-ct}{2}}(C_5 + C_6 t). \qquad (41)$$

for arbitrary C_5 and C_6. This type of damping is called critical damping which give fastest return of the system to its equilibrium position. When $c = 0$, the remaining term is equal to zero and the response is not sinusoid.

Conclusion

Here in this study, we explore the Boundedness and global asymptotic stability of solutions of Diffung equations with damping. The usage of an appropriate Lyapunov second method alongside the eigenvalue approach and the introduction of some necessary parameters such as $K_n : n = 1\,,\ 2\,,\ 3\,,\ \ldots\,,\ 7$ including $p(\tau)$ helped in no small measure, in achieving the desired result. Application of Schartz inequality was also significantly utilized.
The results show that under specific conditions on damping, nonlinear and force terms, the solutions remain remain bounded and converge to an equilibrium or periodic orbit.

One of the gains obtained in the method used in the study is the ability to get the result through the Lapunov method without solving the differential equation fully. The realization of the results also show sharper and more as well as more general conditions for boundedness and stability than the previous results in the existing literature.
Furthermore, numerical solution is also achievable and serves as an excellent alternative to the theoretical or analytic aspect. In Mechanical engineering for example, this result provide much Mathematical support for the long-term behavior of vibrating mechanical systems, in order to guarantee that there would be no unbounded or chaotic motion under normal operating conditions. Essentially, the established conditions for global asymptotic stability serve as tools to ensure reliability and safety.

Besides, the results also lay emphasis on the robustness across various forms of Duffing equation for example forced, unforced, with different nonlinear situations. It also contributes to the understanding of the nonlinear dynamics and differential equations, more elaborately.

References

[1] Salas, A. H. S. , Hernandez J. E. C., Hernandez L. J. M., The Duffing Equation and its Applications in Physics, Journal of Mathematical Problems in Engineering, Article ID 9994967:1 – 13

[2] Nafeh, A. H. , Mook D. T., Nonlinear Oscillation, Wiley, New York (1979)

[3] Yang H., Guo H., He, M.: The prediction of Beinjing urban Distribution Amount using Grey Model and Industrial Structure Analysis. Fifth International Joint Conference on Computational Science and Optimization IEEE. DOI: 10.1109/CSO.2012.47

[4] He, C., Amer, T. S., Tian D., Abolila A. F., Galai, A. A., Controlling the Kinematics of a Spring-Pendulum system using an energy harvesting device, Journal of Low Frequency, Noise, Vibration and Active Control. 41(3), 1231 - 1257 (2022).

[5] Yang, L., Li, Y.: Existence and Global Exponential Stability of Almost Periodic solution for a class of Delay Duffing Equation on Time Scale, Abstract and Applied Analysis, Hindawi Publishing Corporation, 857 - 861 (2014)

[6] Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical system and Bifucation of vector field. Journal of Applied Mathematical Sciences, Springer –Verlag New York (1983)

[7] Jordan, D. W., Smith, P.: Nonlinear Analysis and Differential Equations, Clavendon Press Oxford (1977)

[8] Zeeman, E.: Duffing Equation in Brain Modelling, Bull IMA 12:207 - 214

[9] Oyesanya, M. O..: Duffing Oscillator as a model for predicting earthquake occurrence. Journal of Nigerian Association of Mathematical Physics, 12: 133 - 142 (2008)

[10] Wang, G., Zheng, W.: Estimate of Amplitude and Phase of a Weak Signal by Using the Property of Sensitive Dependency on initial Condition of a Non-linear Oscillator Signal Process 82: 103-115 (2002).

[11] Chen, G., Han, F., Yu, X., Feng, Y.: A further study of Nonlinear feedback system with chaotic oscillation, 1:83-86 (2004)

[12] Fan, J., Zhang, Y. R., Liu, Y., Wang, F., Cao, Q., Yang, T.: Explanation of the Cell Orientation in a Nonofiber Membrane by the Geometric Potential Theory. Results in Physics.

[13] Zi, Y. L., Guo, H. Y., Wen, Z., Yeh, M. H., Hu, Z., Wang, Z. L.: Harvesting Low Frequency ($< 5HZ$) Irregular Mechanical energy: A Possible Killer Application of Triboelectric Nano generator. ACS Nano, 10(4):4797 - 4805.

[14] Li, H. D., Tian, C., Deng, Z. D.: Energy Harvesting fro Low Frequency Application using piezoelectric materials. Journal of Applied Physics Reviews. 1(4): 041301. DOI:10.1063/1.4900845

[15] Njoku F. I., Omari, P.: Stability properties of periodic solutions of a Duffing Equation in the presence of lower and upper solution. Journal of Applied Mathematics and Computer. 135:471 - 490.

[16] Oyesanya, M. O., Nwamba, J. I.: Stability analysis of cubic –quintic Duffing Oscillator: World Journal of Mechanics. 3:299 - 307 (2013)

[17] Tung, C., Tung, O.: A note of Stability and Boundedness of Solutions to Nonlinear Differential Systems of Second Order, Journal of the Association of Arab Universities For Basic and Applied Sciences 24: 169 - 175 (2017).

[18] Jordan D. W., Smith, P.: Nonlinear Analysis and Differential Equations. Clavendon Press Oxford, (1977).

[19] Thomson, J. M. T., Stewart, H. B.: Nonlinear Dynamics and Chaos. John Wiley and Sons, New York (1986)

[20] Nnaji, D. U., Makuochukwu, F. O., Ezenwobodo, S. S., Stguegboh, N., Onyiaji, N. E.: On the Stability of Duffing Type Fractional Differential Equation with Cubic Nonlineariry. Journal of Scientific Research. 7(5): 1 - 14.

[21] Tian, D., Ain, Q. T., Anjum, N.: Fractal N/MEMS from Pull-in Instability to Pull-in Stability. Journal of Fractals, 29(2):2150030 (2020)

[22] Tian, D., He, C. H., He, J. H.: Eractal Pull-in Stabilitytheory for microelectro-mechanical systems. Front Phys, 9:606011 (2020).

[23] Ortega, R., Alonso, J. M.: Boundedness and Asymptotic Stability of a Forced and Damped Oscillator. Nonlinear Analysis Theory, Method and Applications. 25(3): 297 - 309 (1995)

[24] Tung, C., Tung, E.: On the Asymptotic Behaviour of Solutions of Certain Second Order Differential Equations. Journal of the Franklin Institute. 344(5):391 - 398 (2007).

[25] Eze, E. O., Ajah, R. O.: Application of Lyapunov and Yoshizawa's theorems on Stability, Asymptotic Stability, Boundaries and periodicity of solutions Duffing Equation: Asian Journal of Applied Science, 2(6):970 - 975 (2014)

[26] Eze, E. O., Ezeora, J. N., Obasi, U. E.: Astudy of Periodic Solution of a Duffing Equation by Implicit Function Theorem. Open Journal of Applied Sciences, 8(10): 459 - 464 (2016).

[27] Ding, T.: Boundedness of Solutions of Duffing Equations. Journal of Differential Equations, 61(2):178 - 207 (1986)

[28] Zheng, J.: Boundedness and Global asymptotic stability of constant equilibrium in a fully parabolic chemotaxis system with nonlinear logistic source. Journal of Mathematical Analysis and Applications. 450(2):1047 - 1061 (2017)

[29] Valdes, J. E. N.: On the Boundedness and Global Asymptotic Stability of the Lienard Equation with Resorting terms. Revista de la, Union Mathematica Argentina, 41(4):47 – 59.

[30] Laser, A. C., Mekenna, P. J.: On the Existence of Stable Periodic Solutions of Differential Equations of Duffing Type: Proceeding of American Mathematical Society, 110:125 - 133 (1990).

[31] Ortega, R.: The first Interval of Stability of a Periodic Equation of Duffing Type. Proceedings of American Mathematical Society, 115:1061 - 1067 (1992).

[32] Levin, J. J., Nohel, J. A.: Global Asymptotic Stability for Nonlinear Systems of Differential Equation and Application to Reactor Dynamics. Archs ration, Mechanical Analysis: 5; 194 - 211 1960).

[33] Thurtson, L. H., Wong, J. S. W.: On Global Asymptotic Stability of certain second order of certain Second order Differential Equation with Integrable forcing terms. SIAM Journal on Applied Mathematics, 24(1): 50 - 61 (1973)

[34] Baillieu, J., Willems, J. C.: Mathematical Control Theory: Published by Springer-Verlag New York In. Ist Edition

[35] Eze, E. O., Ukeje, E., Hilary, M. O.: Atable Bounded and Periodic Solutions in a Nonlinear Second Order Differential Equations. American Journal of Engineering Research 4(8):14 - 18 (2018)