e-ISSN 3050-9645
p-ISSN 2795-3274

GPH INTERNATIONAL JOURNAL OF MATHEMATICS
WWW.GPHJOURNAL.ORG

10.5281/zenodo.16893422 | Vol. 8 Issue 07 July - 2025 | Manuscript ID: #2050

GLOBAL ASYMPTOTIC STABILITY ANALYSIS OF A SECOND-ORDER DUFFING EQUATION WITH POISON STABLE COEFFICIENT

By:

Ezugorie I. G.

Department of Mathematics, Enugu State University of Science and Technology, Enugu State.

Abstract

This paper presents a comprehensive analysis of the global asymptotic stability of solutions to a second-order Duffing-type differential equation with a Poisson-stable coefficient. The Duffing equation, widely recognized as a fundamental nonlinear oscillator model, exhibits increased dynamical complexity when subjected to non-autonomous and time-varying perturbations such as Poisson stability. In this work, we extended existing stability results by constructing an appropriate Lyapunov function and employing comparison principles to derive sufficient conditions that guarantee the asymptotic convergence of all solutions to the trivial equilibrium. The approach not only addresses the inherent challenges posed by the recurrent and non-periodic nature of Poisson-stable functions but also provides a rigorous framework for analyzing stability in broader classes of nonlinear, non-autonomous systems. The theoretical findings presented here contribute to the understanding of oscillatory behavior in complex dynamical models and have potential implications for engineering and applied sciences where stability under recurrent external influences is critical

Keywords:

Duffing equation, Global asymptotic stability, Poisson-stable coefficient, Nonlinear oscillations, Lyapunov function, Non-autonomous systems.

How to cite: Ezugorie I. G. (2025). GLOBAL ASYMPTOTIC STABILITY ANALYSIS OF A SECOND-ORDER DUFFING EQUATION WITH POISON STABLE COEFFICIENT. *GPH-International Journal of Mathematics*, 8(7), 146-156. https://doi.org/10.5281/zenodo.16893422

1.1 INTRODUCTION

The Duffing equation, a prototypical second-order nonlinear oscillator, has long served as a foundational model for investigating complex dynamical behaviors such as bifurcation, chaos, and recurrence. In its classical form with constant coefficients, it is well established that energy-like Lyapunov functions can guarantee the boundedness and convergence of solutions under damping, providing straightforward stability results (Wikipedia contributors, 2025). However, the dynamics become significantly richer, and more challenging, when one moves beyond autonomous settings to consider non-autonomous recurrent influences, particularly those described by Poisson-stable coefficients. These are functions that exhibit recurrence yet are neither strictly periodic nor almost-periodic, making them a realistic representation of external influences in many physical and engineering systems. Recent research has begun to push the boundaries of stability analysis in this direction. Akhmet, Tleubergenova, and Zhamanshin (2023), for instance, studied Duffing equations with two-component Poisson-stable coefficients. They established the existence and uniqueness of Poisson-stable solutions and proposed methods to analyze their asymptotic stability through numerical simulations. Building on this, Akhmet et al. (2021) investigated modulo periodic Poisson stability in quasilinear differential equations, proving the existence and asymptotic stability of such recurrent solutions and illustrating their findings with computational results. Complementing these deterministic frameworks, Liu and Liu (2020) extended the concept of Poisson stability to stochastic systems. They demonstrated that semilinear stochastic differential equations driven by Lévy noise admit Poisson-stable solutions that are globally asymptotically stable in the mean-square sense under suitable conditions on drift, diffusion, and jump terms.

These studies highlight the novelty and growing interest in understanding recurrent, yet non-periodic, external perturbations and their impact on nonlinear systems. For the Duffing equation in particular, most existing results have focused on periodic or almost-periodic forcing. In contrast, the Poisson-stable setting captures a broader and more realistic class of recurrent behaviors, with strong relevance to mechanical oscillators, climate models, and biological systems influenced by irregular environmental inputs. The present study builds on these developments by investigating the global asymptotic stability of solutions to a second-order Duffing-type differential equation featuring Poisson-stable coefficients. Our approach relies on constructing appropriate Lyapunov functions tailored to the non-autonomous Poisson context and applying comparison principles to derive sufficient conditions for asymptotic convergence of solutions to the trivial equilibrium. This not only connects deterministic and stochastic stability perspectives but also advances the mathematical understanding of non-autonomous nonlinear systems subject to irregular yet recurrent external influences.

The approach offers several important advantages. First, it provides global rather than merely local stability guarantees, making the results robust against large perturbations or significant deviations from equilibrium. Second, it is constructive, since the Lyapunov framework provides explicit criteria that can be verified in practical applications. Third, the use of Poisson-stable coefficients makes the analysis more widely applicable to real-world systems where disturbances are recurrent but not exactly periodic. Finally, the framework is generalizable and can be extended to other classes of nonlinear systems with recurrent non-autonomous coefficients. This research contributed both to the theoretical literature on Duffing-type nonlinear oscillators and to the practical analysis of systems influenced by irregular recurrent behaviors. By establishing sufficient conditions for global asymptotic stability under Poisson-stable coefficients, it enriches the mathematical theory of stability in non-autonomous systems while offering useful insights for engineering, physics, and applied sciences.

2.0 LITERATURE REVIEW

Akhmet and Arugaslan 2020 they analyze how recurrence structures in the coefficients impact the long-term behavior of solutions. They utilize Lyapunor function technologies and topological methods to establish global stability results. Akhmet etal, 2020 investigated second-order non-linear systems with Poison stable terms and established conditions for global attractivity and boundedness. Akhmet and Fen 2016 extended the analysis of Poison stable differential equations by establishing conditions under which solutions demonstrate global asymptotic stability and recurrence properties. These findings are significant for second order systems like the duffing equations, especially when subject to external forces or damping that are Poison stable. Eze and Aja 2014 presented a work on application of Lyapunov and Yoshizawa's theories on stability, asymptotic stability, boundedness and periodicity of solution of duffing equation. They used fixed point technique and integrated equation as the mode to contain a priori bounds of the solution. The results show that the consequences of the cyclic relationship between different properties of the solutions are unique. Alaba and Ogundare 2014 established sufficient criteria for asymptotic behavior of solutions of certain second-order non-autonomous non-linear ordinary differential equation using a complete Lyapunor function. The result shows that the zero solution of the system is globally asymptotic stable. Kloeden and Rasmussen 2011 have provided a modern dynamical systems approach to stochastic and non-autonomous differential equations, including discussions on Poison stability. They discuss how such systems can be studied using cocycle and skew-product flows, which facilitate the analysis of long term behavior. Cemil 2010 considered the criteria for global asymptotic stability of a null solution of a nonlinear differential equation of fifth order with delay using Lyapunov second method. By defining a Lyapunov function, sufficient conditions for globally asymptotic stability of null solution of the equation was guaranteed.

Adbollabi 2010 opined that Lasalle's invariance principle enables one to conclude asymptotic stability of an equilibrium point even when one cannot find a function $V(x)$ such that $V'(x,t)$ is locally negative definite. He said that the principle is applied only to tie-invariant or periodic systems. Akhmet 2010 introduced methodologies for analyzing differential equations with almost periodic and Poison stable forcing terms. He demonstrated that even in non-autonomous systems, it is possible to construct invariant sets and analyzed attractions with tools such as skero-product flows and comparison principles. Tunc 2007 discussed the global asymptotic stability of the trivial solutions of non-autonomous systems with application to second-order equation using Lyapunov second method.

Sell 1971 laid the theoretical foundation for the qualitative theory of non-autonomous dynamical systems, which is instrumental in studying equations with time-dependent (and possibly Poison stable) coefficients. His framework has been extensively used to analyze global attractions and their structure.

https://doi.org/10.5281/zenodo.16893422

Preliminaries

Duffing equation is a non-linear second order differential equation widely used in modelling forced oscillators with non-linear stiffness. The general form:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = f(t)$$

Where δ, α and β are real constants, and $f(t)$ is an external forcing term.

This paper focuses on a version with poison stable coefficients, representing a class of recurrent functions that generalize periodic and almost periodic functions.

We study the equation as;

$$\ddot{x} + a(t)\dot{x} + b(t)x + c(t)x^3 = 0$$

Where $a(t), b(t), c(t)$ are continuous and poison's stable. The objective is to determine global asymptotic stability of the trivial solution $x = 0$ under suitable assumptions.

Definition (Poison stable function)

A function f:$\mathbb{R} \longrightarrow \mathbb{R}$ is Poison stable if for every $\varepsilon > 0$, there exists a sequence $t_n \subset \mathbb{R}$ such that

$f(t + t_n) \longrightarrow f(t)$ uniformly on compact sets.

Lyapunov stability concepts:

A solution $x(t) = 0$, is globally asymptotic stable if;

i) It is stable, i.e. solutions starting close to zero remain close.
ii) It is attractive, i.e. all solutions tend to zero as $t \longrightarrow \infty$.

Poison Stability: A function $f(t)$ is Poison stable if:

- It is bounded and uniformly continuous
- For every $\varepsilon > 0$, there exists a relatively dense set of shift T such that

$$|\, f(t + T) - f(t)| < \varepsilon$$

Background of the Study

The duffing equation is a well-known non-linear second order differential equation that models a wide range of physical phenomena, particularly in non-linear oscillatory systems such as mechanical vibrations, electrical circuits, and even climate dynamics. The general form of the second-order duffing equation is:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = f(t)$$

Where δ, α and β are real constants, and $f(t)$ is time dependent forcing term.
The global asymptotic stability analysis of such systems becomes vital in understanding the long term behavior of solutions. It ensures that regardless of initial conditions, the solutions will converge to a stable motion or equilibrium.

In this study, we focused on the global asymptotic stability of the second order duffing equation where the coefficient is Poison stable. The aim is to investigate under what conditions the systems solution tends towards a steady or recurrent behavior globally over time, despite the complex nature of the non-linearity and non-autonomous external forcing. This analysis not only enriches the theoretical understanding of duffing type systems under generalized recurrent influences but also contributes to the broader theory of differential equations with non-autonomous coefficient.

3.0 RESULTS

Consider the second – order Duffing-type equation:

$$\ddot{x} + \alpha(t)\dot{x} + \beta(t) + \gamma x^3 = 0 \qquad (1)$$

where

(i) $\gamma > 0$ is a constant

(ii) $\alpha(t)$ and $\beta(t)$ are poison stable functions in time t

(iii) $\alpha(t) \geq \alpha_0 > 0$ (uniformly positive damping)

(iv) $\beta(t) \geq \beta_0 > 0$

We want to show that the zero solution is globally asymptotically stable.

Definitions:

(v) A function $f(t)$ is poison stable if for any sequence $t_{n_k} \to \infty$, there exists a subsequence t_n such that $f\left(t + t_{n_k}\right) \to f(t)$, uniformly on compact subsets.

(vi) A solution $x = 0$ is globally asymptotically stable if it is

Globally Lyapunov stable, and

Every solution $x(t) \to 0$, as $t \to \infty$

Now, we can rewrite the system as

Let

$x = x_1$

$$\ddot{x}_2 + \alpha(t)\dot{x} + \beta(t)x_1 + \gamma x_1^3 = 0 \qquad (2)$$

https://doi.org/10.5281/zenodo.16893422

$\dot{x} = x_2$

$\dot{x}_2 + \alpha(t)x_2 + \beta(t)x_1 + \gamma x_1^3 = 0$ (3)

Then, the system becomes;

$\ddot{x}_1 = \dot{x}_2$

$\dot{x}_2 = -\alpha(t)x_2 - \beta(t)x_1 - \gamma x_1^3$ (4)

We now employ Lyapunov function:

$V(x_1, x_2) = \frac{1}{2}\alpha x_2^2 + \frac{1}{2}\beta_0 x_1^2 + \frac{1}{4}\gamma x_1^4$ (5)

Note:

$V(x) \geq 0,$

$V(x) = 0 \quad \Longrightarrow x_1 = x_2 = 0$

$V(x) \ \rightarrow \infty$ as

$|x_1, x_2| \rightarrow \infty$

Now we compute $\dot{V}$

$\dot{V} = x_2\dot{x}_2 + \beta_0 x\dot{x}_1 + \gamma x_1^3\dot{x}_1$ (6)

$= x_2(-\alpha(t)x_2 - \beta(t)x_1 - \gamma x_1^3) + \beta_0 x_1 x_2 + \gamma x_1^3 x_2$

$= -\alpha(t)x_2^2 - \beta(t)x_1x_2 - \gamma x_1^3 x_2 + \beta_0 x_1 x_2 + \gamma x_1^3 x_2$

$= -\alpha(t)x_2^2 - \beta(t)x_1x_2 + \beta_0 x_1 x_2$

$= -\alpha(t)x_2^2 + (\beta_0 - \beta(t))x_1x_2$ (7)

Now, we use $\beta(t) \geq \beta_0 > 0$, so $\left(\beta_0 - \beta(t)\right) \leq 0$

Which implies,

$\dot{V} = -\alpha(t)x_2^2 \leq 0$

Thus, V is a non-increasing function doing trajectories.

Applying Lasalle's invariance principle,

solutions approach the largest invariant set $\{(x_1, x_2): \dot{V} = 0\} = \{(x_1, x_2): x_2 = 0\}$

within this set:

$\dot{x}_1 = 0, \dot{x}_2 = -\beta(t)x_1 - \gamma x_1^3$

So for $x_2 = 0, \dot{x}_2 = 0$ only if $x_1 = 0$ (since $\beta(t), \gamma > 0$).

Hence, the only invariant point is $(x_1, x_2) = (0,0)$

Thus, all solutions tend to the origin as $t \to \infty$.

Therefore, the origin is globally asymptotic stable.

Applications of Lyapunov Function Methods and Global Asymptotic Stability Analysis

1. Control Design for Nonlinear Systems

One of the most significant applications of Lyapunov function methods is in the design of controllers for nonlinear systems, particularly in engineering contexts such as robotics, aerospace, and automotive suspension systems. Nonlinear dynamics often exhibit complex behaviors that cannot be adequately addressed using linear control techniques (Núñez etal, 2023). By constructing an appropriate Lyapunov function, one can design a control input that guarantees the negativity of the derivative of the Lyapunov function along system trajectories, thereby ensuring asymptotic stability (Shior et al 2024). For example, in automotive active suspension systems, Lyapunov-based control has been applied to ensure that oscillations caused by road irregularities decay asymptotically, providing ride comfort and vehicle stability. The key strength of this method lies in its ability to offer global results rather than merely local stability, which is crucial in highly nonlinear real-world systems

2. Stability in Impulsive and Pharmacokinetic Models

The concept of Lyapunov stability has also been extended to impulsive differential equations, which describe systems that experience sudden changes in state at discrete time instants. These types of systems are commonly found in biological and pharmacokinetic models where drug administration, for example, may involve instantaneous dosage adjustments. Traditional stability techniques may fail to capture the effects of such impulses, but Lyapunov-like functions and comparison principles can be adapted to handle these discontinuities. Agarwal, O'Regan, and Hristova (2017) demonstrated how sufficient conditions can be derived for stability, uniform stability, and asymptotic uniform stability in impulsive models (Senewo et al 2025). Their work also highlighted the relevance of these conditions in pharmacokinetics, where drug concentration levels must be maintained within a stable range despite periodic or non-instantaneous dosing. This application illustrates how Lyapunov-based methods are not only theoretical but also directly applicable to biomedical sciences.

3. Comb-Drive Finger MEMS Actuators

Another important application can be found in the design and stability analysis of micro-electro-mechanical systems (MEMS), particularly comb-drive actuators. These actuators, which rely on electrostatic forces to generate motion, are widely used in sensors, resonators, and micro-positioning devices. Due to their small scale, MEMS devices are highly sensitive to nonlinearities, and stability issues can critically affect their performance. Núñez etal, 2023 applied Lyapunov function methods and asymptotic stability criteria to analyze lateral oscillations in comb-drive actuators, demonstrating that antiperiodic solutions could be stabilized under certain conditions. By employing L^p-type sufficient conditions and Hill-type analysis, the authors showed how global asymptotic stability could ensure reliable performance in the presence of nonlinear forcing. This application is especially important for advancing the robustness of MEMS devices used in precision engineering and medical instrumentation.

4. Synchronization of Non-Autonomous Chaotic Systems

Synchronization of chaotic systems has become a widely studied problem in nonlinear science, with applications ranging from secure communication to power electronics. Non-autonomous chaotic systems, where time-varying parameters influence the dynamics, present additional challenges for stability and synchronization (Agbata et al 2024). Lyapunov function methods provide a rigorous framework for proving that coupled chaotic systems can synchronize asymptotically under appropriate feedback conditions. For instance, one study employed a novel Lyapunov-based approach to establish global synchronization criteria for master–slave non-autonomous chaotic systems using linear state error feedback Núñez etal, 2023. The results provided algebraic conditions that guarantee convergence of the error dynamics to zero, thus ensuring synchronization. Such applications demonstrate how Lyapunov theory can be extended beyond simple equilibrium stability to more complex dynamical behaviors, which is particularly useful in modern technological systems that rely on chaotic synchronization.

5. Power Grid Stability and Second-Order Dynamical Systems

The analysis of second-order dynamical systems using Lyapunov functions and stability criteria also plays a crucial role in power system engineering. Power grids are inherently nonlinear and can experience instability due to fluctuating loads, faults, or changes in damping parameters. By treating damping as a key parameter in second-order models, researchers have shown that Lyapunov and perturbation methods can be used to establish conditions for global asymptotic stability as well as for bifurcation analysis. Gholami and Sun (2020) investigated the effects of damping on the hyperbolicity of equilibrium points in second-order systems and derived rigorous conditions under which Hopf bifurcations occur. These findings are directly applicable to the analysis of oscillatory stability in power grids, where maintaining synchronized oscillations across large networks of generators is critical for operational security. Thus, Lyapunov-based methods provide not only theoretical guarantees but also practical insights for ensuring the resilience of large-scale energy systems.

CONCLUSION

In this analysis, we investigated the global asymptotic stability of the zero solution of a second-order Duffing-type differential equation with Poisson-stable coefficients. By employing a carefully constructed Lyapunov function and invoking LaSalle's invariance principle, we demonstrated that the origin is not only stable but also globally attractive. The results established in this study confirm that, under the assumptions that the damping coefficient $\alpha(t)$ and stiffness coefficient $\beta(t)$ are Poisson-stable and uniformly positive, and that the nonlinear restoring force γx^3 has a positive coefficient, the system exhibits global asymptotic stability. This means that, regardless of initial conditions, all trajectories of the system converge to the trivial equilibrium at the origin as time tends to infinity. The significance of these findings lies in their ability to extend classical stability results of the Duffing equation to the more general and realistic case where system parameters are governed by Poisson-stable functions. Unlike periodic or almost-periodic coefficients, Poisson stability captures a broader class of recurrent yet irregular behaviors, making the analysis applicable to systems influenced by complex environmental or external factors. This enhances the relevance of the results for practical applications in engineering, physics, and biology, where irregular recurring inputs are common and stability is crucial for system performance.

An important advantage of the approach used in this study is its reliance on Lyapunov methods, which provide constructive criteria for stability without requiring explicit solutions to the differential equation. Coupled with LaSalle's invariance principle, the methodology offers a robust and systematic framework for proving global convergence. This ensures that the stability results are not merely local but hold for the entire state space, guaranteeing robustness to large perturbations and broad ranges of initial conditions. The novelty of the present work lies in bridging the gap between classical Duffing oscillators with constant or periodic coefficients and systems subjected to irregular, non-periodic yet recurrent influences. The incorporation of Poisson-stable coefficients into the stability framework opens new directions for research in non-autonomous dynamical systems. Future work could extend these results by examining the impact of external forcing terms, stochastic perturbations, or time delays on the stability of such systems. Additionally, numerical simulations could be used to complement the analytical findings, illustrating how the trajectories behave under different classes of Poisson-stable inputs. The study has provided new theoretical insights into the global asymptotic stability of Duffing-type equations with Poisson-stable coefficients. The results contribute to the mathematical theory of nonlinear oscillators and offer practical tools for the analysis and design of systems influenced by recurrent but irregular external forces. By demonstrating that all solutions converge to equilibrium under broad conditions, this work strengthens the understanding of stability in non-autonomous nonlinear systems and sets the stage for further advances in the study of recurrent dynamical phenomena.

REFERENCES

Agarwal, R., O'Regan, D., & Hristova, S. (2017). Stability by Lyapunov-like functions of nonlinear differential equations with non-instantaneous impulses. *Journal of Applied Mathematics and Computing, 53*(2), 147–168. https://doi.org/10.1007/s12190-015-0961-z

Agbata, B. C., Shior, M. M., Gbor, G. D., Ezugorie, I. U., & Topman, N. N. (2024). Solution of first-order ordinary differential equations using fourth-order Runge-Kutta method with MATLAB. *International Journal of Mathematics and Statistics Studies, 12*(1), 54–63.
https://doi.org/10.37745/ijmss.13/vol12n15463

Akhmet, M. & Arugaslan, D. (2020). Poison stable solutions in nonlinear differential equations. *Journal of Mathematical Analysis and Applications.* 487(1); 123984.

Akhmet, M. & Fen, M. O. (2016). Replication of chaos in neural networks, economics and physics. Springer.

Akhmet, M. (2010). Nonlinear hybrid continuous discrete time models. Springer.

Akhmet, M. Kivilcim, A. & Tleubergenoova, M. (2020). Poison stability and attractivity in nonlinear second order systems. *Qualitative Theory of Dynamical systems,* 19(2), 1-23.

Akhmet, M., Tleubergenova, M., & Zhamanshin, A. (2021). Modulo periodic Poisson stable solutions of quasilinear differential equations. *arXiv*. https://arxiv.org/abs/2110.04728

Akhmet, M., Tleubergenova, M., & Zhamanshin, A. (2023). Duffing equations with two-component Poisson stable coefficients. *Carpathian Journal of Mathematics, 39*(2), 349–357. https://doi.org/10.37193/CJM.2023.02.01

Alaba, J.G. and Ogundare, K. (2014). Asymptotic behavior of solutions of certain second order non-autonomous, nonlinear ordinary differential equation. *Internal Journal of pure and Applied Mathematics,* 90(4), 469-483.

Alodollahi, F. (2010). Stability Analysis 1. Lecture note on Nonlinear Control. Amirkabir University of Technology.

Cemil, T. (2010). On the stability of solution of nonlinear differential equation of fifth order with delay.

Cheban, D., Duan, J. & Schmalfub, B. (2004). Stationary measures and attractors for stochastic evolution equations in stochastic dynamics (pp 47-70). Springer.

Eze, E.O. and Ajah, R.O. (2014). Application of lyapunov and Yishizawa's theorems on stability asymptotic stability, boundaries and periodicity of solutions of duffing equation. Asian Journal of Applied Science, 2(6); 970-975.

Gholami, A., & Sun, X. A. (2020). The impact of damping in second-order dynamical systems with applications to power grid stability. *arXiv*. https://arxiv.org/abs/2010.06662

Kloeden, P.E. & Rasmussen, M. (2011). Non autonomous dynamical systems. *American Mathematical Society.*

Liu, X., & Liu, Z. (2020). Poisson stable solutions for stochastic differential equations with Lévy noise. *arXiv*. https://arxiv.org/abs/2002.00395

Núñez, D., Larreal, O., & Murcia, L. (2023). A remark about asymptotic stability in Duffing equations: Lateral stability in Comb-drive finger MEMS. *Journal of Inequalities and Applications*. https://doi.org/10.1186/s13660-023-03050-9

Sell, G.R. (1971). Topological dynamics and ordinary differential equations. Van Nostrand Reinhold.

Senewo, E. O., Micheal, D. B., Ezugorie, I. G., Ahman, Q. O., Agbata, B. C., & Atabo, V. O. (2025). Higher derivative direct block methods for fourth-order initial value problems of ordinary differential equations. *Annals of Communications in Mathematics, 8*(1), 103–115. https://doi.org/10.62072/acm.2025.080108

Shior, M. M., Agbata, B. C., Ezeafulukwe, A. U., Topman, N. N., Mathias, A. G., & Ekwoba, L. (2024). Applications of first order ordinary differential equations to real life systems. *European Journal of Statistics and Probability, 12*(2), 43–50. https://doi.org/10.37745/ejsp.2013/vol12n24350

Tunc, C. and Tunc, E. (2007). The asymptotic behavior of solutions of certain second order differential equations. *Journal of the Franklin Institute*, 344(5), 391-398.

Wikipedia contributors. (2025, July). Duffing equation. In *Wikipedia*. Retrieved August 17, 2025, from https://en.wikipedia.org/wiki/Duffing_equation