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Infinite Abelian group extracted from an infinite sequence

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Abstract

In this paper, we introduce a reversed symmetric Tribonacchi sequence and establish a new recurrence relation associated with it. We construct an infinite series involving binomial coefficients derived from the classical Tribonacchi sequence, leading to the formulation of an infinite Abelian group. Furthermore, we develop a set of 2 by 2 matrices, forming a matrix subgroup by employing the concepts of eigenvalues and eigenvectors tied to the reversed symmetric Tribonacchi sequence. Our results include closed-form expressions and combinatorial representations for the sums of terms in these newly defined sequences. Finally, we explore the interrelationships among these sequences, demonstrating how they naturally give rise to algebraic group and subgroup structures.

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Introduction:

Semigroup, Monoid and Group

If G is a nonempty set, a binary operation on G is a function G x G $\rightarrow$ G.

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Definition 1: A *semigroup* is a nonempty set G together with a binary operation on G, which is

(i) associative: $a(bc) = (ab)c,\ \forall\, a, b, c \in G$.

A *monoid* is a semigroup G which contains a

(ii) (two sided) identity element $e \in G$ such that $ae = ea = a \forall\, a \in G$.

A *group* is a monoid G such that

(iii) for every $a \in G$ there exist a(two-sided) inverse element $a^{-1} \in G$ such that

$$a^{-1}a = aa^{-1} = e$$

A semigroup G is said to be *abelian* or *commutative* if its binary operation is

(iv) commutative: $ab = ba \forall\, a, b \in G$.

Examples:

1. Let $= \mathbb{R}\backslash\{0\}$, and $n \in \mathbb{N}$. Then G= $GL(n, F)$ be the set of all n by n matrice with entries in F. this is a group with $g * h$ given by matrix multiplication.
2. Symmetric groups (Symmetries of Graphs): We view n-*gon* as a graph of n sides. The symmetries of an n- gon form a group.

We sometimes can proof some facts just by providing a counter example. To provide familiar counter examples or to construct such an example from a given set is not an easy task. Group theory's origin trace back to the late 18th and 19th centuries, with key contributions from Lagrange, Ruffini, and Abel in the context of solving polynomial equations. Evariste Galois is created with formalizing the concept of a group and its connection to the solvability of equations, now known as Galois Theory. Geometry and number theory also played a significant role in its development.

Group theory continued to develop through the 20th century, becoming a cornerstone of abstract algebra and finding applications in diverse areas of mathematics and physics. A major achievement of group theory in the late 20th century and early 21st centuries was the classification of finite simple groups.

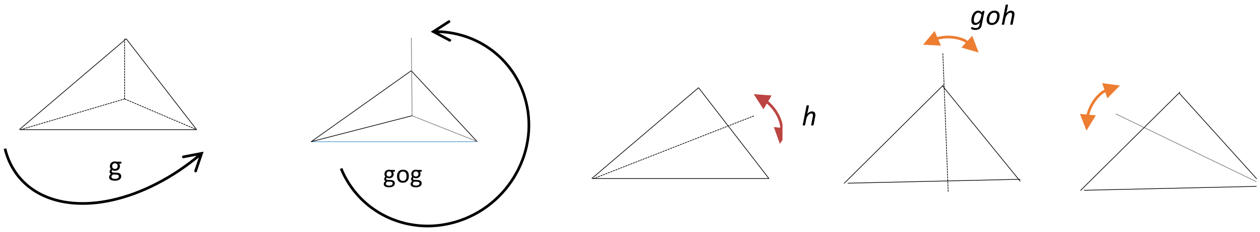
For instance

We view n-*gon* as a graph of n sides. The symmetries of an n- gon form a group.

A. B. C. D.

Note: Rotation anticlockwise by $\frac{2\pi}{3}$ from each vertex, and reflection about the line through each vertex of a triangle describes the symmetries of an equilateral triangle.

Symmetries of an equilateral triangle. Consider a 3-gon that is an equilateral triangle There are precisely six symmetries of the 3-gon.



$D_3 = \{e, g, gog, h, goh, hog\} = \{e, g, g^2, h, gh, hg\}$. The identity e is not included in the above diagram here.

Rotation anticlockwise by 2π/3 (which we call g) and reflection about the median

The Dihedral Group

The dihedral group D_n is the group consisting of the Rotation and Reflection of an n – sided regular polygon that transform the polygon into itself. For such a polygon, we can rotate it by $2\pi n$ about its center, or reflect it about a line of symmetry that passes through its center.

The group element corresponding to a rotation by $2\pi n$ is denoted by r. Repeating the rotation gives the elements

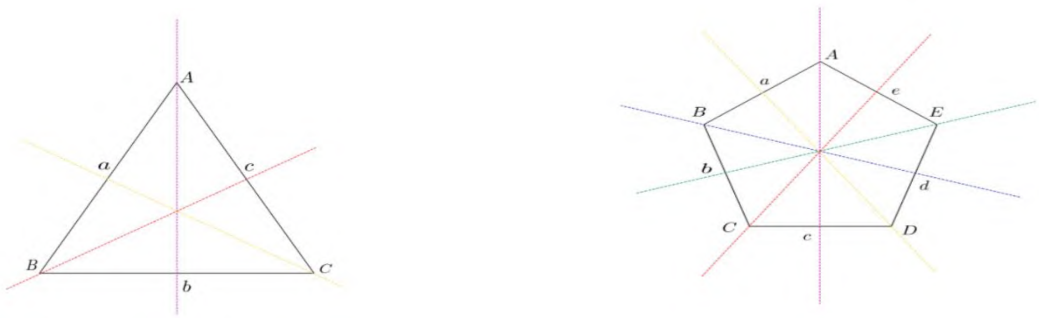
$$\left\{2\left(\frac{2\pi}{n}\right), 3\left(\frac{2\pi}{n}\right), 4\left(\frac{2\pi}{n}\right), ---, n\left(\frac{2\pi}{n}\right) = 2\pi\right\}$$

Since applying , n times restores the polygon to its original orientation, we have r$n = 1$. Thus , counting the identity, there are n rotation elements in the dihedral group D_n.

What about reflections? Since we can reflect about any of the n medians, there are n distinct reflections. Thus the order of D_n is $|D_n| = 2n$.

If we denote a reflection by S_i , for $i = 1, 2, 3, ---, n$ then we see geometrically that repeatings_i returns the polygon to its original state, so ${s_i}^2 = e$.

Thus D_n= $\{1, r, r^2, r^3, -, -, -, r^{n-1}, s_1, s_2, s_3, ---, s_n\}$ form a group



Sequence is a list of numbers arranged in a specific order. It can contain members similar to a set. However, sequence can have the same members repeated as much as possible at divergent locations. Thus, pattern is a substantial element of a sequence. For instance, arithmetic, geometric, square, cube, Lucas, Fibonacci and Tribonacchi.

Definition 2:

A symmetric Inversed sequence: A symmetric Inversed sequence is a sequence of numbers that has the property of set of Integers.

$$\mathbb{Z} = \{- - -, -4\ , -3\ , -3, -1\ , 0\ , 1\ , 2\ , 3\ , 4\ , - - -\}$$

$$= \{- - -, -4\ , -3\ , -3, -1\ \} \cup \{0\} \cup \{1\ , 2\ , 3\ , 4\ , - - -\}$$

$$= \mathbb{Z}^{-} \cup \{0\} \cup \mathbb{Z}^{+}$$

$$= \mathbb{Z}_{-n} \cup \{0\} \cup \mathbb{Z}_{n} (\text{Define: } \mathbb{Z}_{-n} = -\mathbb{Z}_{n})$$

The$\mathbb{Z}$ set of integer, $(\mathbb{Z}\ , +\ )$ is an example of an infinite set of commutative group.

Fibonacci sequence is a sequence starting from 0 and 1 where the succeeding terms are taken from two previous terms that are stated. Moreover, Fibonacci – like sequences is a derivative of the Fibonacci sequence where the same pattern is applied. There only difference is that Fibonacci –like sequences starts at any two given terms.

On the other hand, Tribonacchi sequence is a sequence of whole numbers with equations $T_0 = 0$, $T_1 = 0$, and $T_2 = 1$, in which $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. This only means that the previous three terms are added to find the next terms.

Just like Fibonacci and Fibonacci –like sequence, when the first three terms of the Tribonacchi sequences become arbitrary, it is known as Tribonacchi-like sequence. These sequences have many applications in environment, biology, and chemistry. Arts. Mathematics, music and among others as it occurs naturally.

Definition3

A Tribonacchi-like sequence is a sequence of number defined by
$T_0 = 0$; $T_1 = 1$ and $T_2 = 1$
and the recurrence equation

$T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. For each Tribonacchi-like sequenceT_n,

$G_n^i = \sum_{k=0}^{i} (-1)^k \binom{i}{k} T_{n+i-k}$, and $F_n^i = \sum_{k=0}^{i} \binom{i}{k} T_{n+i-k}$ are always

Tribonacchi- like sequences.

Definition 4:

Tribonacchi Sequence: Tribonacchi sequence is a sequence of whole numbers with equations

$T_0 = 0$; $T_1 = 1$ and $T_2 = 1$

in which $T_n = T_{n-1} + T_{n-2} + T_{n-3}$.

When the first three terms of the Tribonacchi sequence become arbitrary, it is known as Tribonacchi-like sequence. Tribonacchi like sequence can start at any desired number.

Definition 5:

The sequence
$S_1, S_2, S_3, \ldots S_n$ in which $S_n = T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-1}S_3$

was obtained in solving the nth term of Tribonacchi-like sequence using the first three terms
S_1 ,S_2 and S_3 and Tribonacchi numbers and is called is a generalized Tribonacchi Sequence.

These sequences have many applications in environment, Biology, Chemistry, Arts, Mathematics, Music and among others as it occurs naturally.

The first 20 Tribonacchi Numbers. See the table below

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
0	1	1	2	4	7	13	24	44	81

T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}
149	274	504	927	1705	3136	5768	10609	19513	35890

The Tribonacchi numbers are a generalization of the Fibonacci defined by

$T_0 = 0$; $T_1 = 1$ and $T_2 = 1$
and the recurrence equation
(1) $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$.

In these patterns, a formula
(2) $S_n = T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-1}S_3$ was obtained in solving the nth term of Tribonacchi-like sequence using the first

Three terms S_1 , S_2 and S_3 and Tribonacchi numbers.

Theorem 1:

$T_0 = 0, \quad T_1 = 1 , \; T_2 = 1$

$T_n = T_{n-3} + T_{n-2} + T_{n-1} \;, \quad \forall\, n \geq 3$

Define: $S_n = {T_{n+1}}^2 - {T_n}^2 = G_n F_n$ where $G_n = T_{n+1} - T_n$ and

$$F_n = T_{n+1} + T_n$$

G_n and F_n are Tribonacchi-like sequences.

Proof: I. Set $G_n = T_{n+1} - T_n$, $\forall\, n \geq 3$ .We show that $G_n = G_{n-3} + G_{n-2} + G_{n-1}$.

$G_n = T_{n+1} - T_n = T_{n-2} + T_{n-1} + T_n - T_n = = T_{n-2} + T_{n-1}$

$= (T_{n-3} + T_{n-2} + T_{n-1})$ - T_{n-3}

$= T_n - T_{n-3}$

Consider $G_{n-3} = T_{n-2} - T_{n-3}$ (1)

$G_{n-2} = T_{n-1} - T_{n-2}$ and

$G_{n-1} = T_n - T_{n-1}$

Combining the above, we've $G_{n-3} + G_{n-2} + G_{n-1}$

$=(T_{n-2} - T_{n-3}) + (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$

$= T_n - T_{n-3}$ (2)

From (1) and (2) $G_n = G_{n-3} + G_{n-2} + G_{n-1}, \forall \geq 3$

II. Set $F_n = T_{n+1} + T_n$, $\forall\, n \geq 3$. We show that $F_n = F_{n-3} + F_{n-2} + F_{n-1}$.

$F_n = T_{n+1} + T_n = T_{n-2} + T_{n-1} + T_n + T_n = T_{n-2} + T_{n-1} + 2T_n$

$= T_{n-2} + T_{n-1} + 2(T_{n-3} + T_{n-2} + T_{n-1})$

$= 3T_{n-2} + 3T_{n-1} + 2T_{n-3}$

$= 3T_{n-2} + 3T_{n-1} + 3T_{n-3}$ - T_{n-3}

$= 3(T_{n-2} + T_{n-1} + T_{n-3})$ - T_{n-3}

$= 3T_n - T_{n-3}$ (3)

Consider the sequence

$F_{n-3} + F_{n-2} + F_{n-1} = (T_{n-2} + T_{n-3}) + (T_{n-1} + T_{n-2}) + (T_n + T_{n-1})$

$= 2T_{n-2} + 2T_{n-1} + T_{n-3} + T_n$

$= 2T_{n-2} + 2T_{n-1} + T_{n-3} + T_n + T_{n-3} - T_{n-3}$

$= 2(T_{n-2} + T_{n-1} + T_{n-3}) + T_n - T_{n-3}$

$= 3T_n - T_{n-3}$ (4)

from (3) and (4) we've $F_n = T_{n+1} + T_n$ is Tribonacci –like sequences. ⊡

Consider the subsequences of Tribonacchi –like sequences F_n and G_n. For conveniences we denote the first subsequences of F_n and G_n by $F_n{}^1$ and $G_n{}^1$ respectively, so that

$\boldsymbol{F_n{}^1 = T_{n+1} + T_n}$ and $\boldsymbol{G_n{}^1 = T_{n+1} - T_n}$ and the Second, the Third, . . . , and the $(t+1)th$ Tribonacchi-like subsequences respectively

$F_n{}^2 = F_{n+1}{}^1 + F_n{}^1$, $G_n{}^2 = G_{n+1}{}^1 - G_n{}^1$

$F_n{}^3 = F_{n+1}{}^2 + F_n{}^2$; $G_n{}^3 = G_{n+1}{}^2 + G_n{}^2$

⋮ ⋮

$F_n{}^{t+1} = F_{n+1}{}^t + F_n{}^t$, $G_n{}^{t+1} = G_{n+1}{}^t - G_n{}^t$

We will explore the subsequence of $G_n{}^1 = T_{n+1} - T_n$. (5)

Coefficient in orders: (1 , -1)

$G_n{}^2 = G_{n+1}{}^1 - G_n{}^1 = (T_{n+2} - T_{n+1}) - (T_{n+1} - T_n)$

$= T_{n+2} - 2T_{n+1} + T_n$ (6)

Coefficients in order: (1 , -2 , 1)

$G_n{}^3 = G_{n+1}{}^2 - G_n{}^2 = (T_{n+3} - 2T_{n+2} - T_{n+1}) - (T_{n+2} - 2T_{n+1} + T_n)$ (from (5))

$= T_{n+3} - 3T_{n+2} + 3T_{n+1} - T_n$ (7)

Coefficient in order: (1, -3,3, -1)

$G_n{}^4 = G_{n+1}{}^3 - G_n{}^3 = (T_{n+4} - 3T_{n+3} + 3T_{n+2} - T_{n+1}) - (T_{n+3} - 3T_{n+2} + 3T_{n+1} - T_n)$
(From (7))

$$= T_{n+4} - 4T_{n+3} + 6\,T_{n+2} - 4T_{n+1} + T_n \qquad (7)$$

Coefficient in order: (1 , -4 , 6 , - 4, 1)

Coefficients are in the sequence of Pascal Triangle but the sign.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Similarly, one can show that

$F_n{}^1 = T_{n+1} + T_n$

$F_n{}^2 = F_{n+1}{}^1 + F_n{}^1 = T_{n+2} + 2T_{n+1} + T_n$

$F_n{}^3 = F_{n+1}{}^2 + F_n{}^2 = T_{n+3} + 3T_{n+2} + 3T_{n+1} + T_n$

$\vdots \qquad\qquad \vdots$

$F_n{}^4 = F_{n+1}{}^3 + F_n{}^3 = T_{n+4} + 4T_{n+3} + 6\,T_{n+2} + 4\,T_{n+1} + T_n$

Theorem 2:

Define $G_n^i = \left(\Delta^{(i)}\,\boldsymbol{T}\right)_{\boldsymbol{n}} = \sum_{k=0}^{i}(-1)^k \binom{i}{k} T_{n+i-k}$, (*)

and

$F_n^i = \left(\Delta^{(i)}\,\boldsymbol{T}\right)_{\boldsymbol{n}} = \sum_{k=0}^{i} \binom{i}{k} T_{n+i-k}$ (**)

G_n^i and F_n^i are subsequences of Tribonacchi-like sequences for $\forall\, i \geq 1$ and $\forall\, n \geq 3$.

That is
$\left\{G_n^{(i)}\right\}_{n\geq 3}$ is a Tribonacci sequence in the variable n .

where $T_0 = 0, \quad T_1 = 1, \; T_2 = 1$

$T_n = T_{n-3} + T_{n-2} + T_{n-1} \quad , \qquad \forall\, n \geq 3.$

Proof:

We prove by induction where $G_n^i = G_{n+1}^{i-1} - G_n^{i-1}$

Let $P(i)$ be the statement that

$P(i): \; G_n^i = \sum_{k=0}^{i} (-1)^k \binom{i}{k} T_{n+i-k}$. (I)

We verify that $P(2)$ is true. When i = 2, the left-side of (1),

That is $G_n^2 = \sum_{k=0}^{2} (-1)^n \binom{2}{k} T_{n+2-k}$ (II)

$\sum_{k=0}^{2} (-1)^n \binom{2}{k} T_{n+2-k} = (-1)^0 \binom{2}{0} T_{n+2-0} + (-1)^1 \binom{2}{1} T_{n+2-1} + (-1)^2 \binom{2}{2} T_{n+2-2}$

$= T_{n+2} - 2T_{n+1} + T_n$ and

$G_n^2 = G_{n+1}^1 - G_n^1$

$= (T_{n+2} - T_{n+1}) - (T_{n+1} - T_n)$

$= T_{n+2} - 2T_{n+1} + T_n$ (III)

From (2) and (3) we see both sides of equation (1) are equal. Hence $P(2)$ is true.

Suppose $P(t)$ is true for some $t \in \mathbb{Z}^+, t \geq 2$, and $\forall\, n \geq 1$ i.e.,

$P(t): \; G_n^t = \sum_{k=0}^{t} (-1)^k \binom{t}{k} T_{n+t-k}$ (IV)

Next, we show that

$P(t+1): \; G_n^{t+1} = \sum_{k=0}^{t+1} (-1)^k \binom{t+1}{k} T_{n+t+1-k}$ (V)

Consider the left-handside $G_n^{t+1} = G_{n+1}^t - G_n^t$

Where $G_{n+1}^t = \sum_{k=0}^{t} (-1)^k \binom{t}{k} T_{n+1+t-k}$ and $G_n^t = \sum_{k=0}^{t} (-1)^k \binom{t}{k} T_{n+t-k}$

Combining the former and later we have

$G_n^{t+1} = G_{n+1}^t - G_n^t$

$= \sum_{k=0}^{t}(-1)^k \binom{t}{k} T_{n+1+t-k}$ - $\sum_{k=0}^{t}(-1)^k \binom{t}{k} T_{n+t-k}$ from (5)

$$= \sum_{k=0}^{t}(-1)^k \binom{t}{k} (T_{n+1+t-k} - T_{n+t-k}) \qquad \text{(VI)}$$

From the binomial expansion $\forall\ k \geq 1$,

$\binom{t+1}{k} = \binom{t}{k} + \binom{t}{k-1}$ and

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!\,(n-k)!}, & 0 \leq k \leq n \\ 0 \quad , & otherwise \end{cases}$$

Hence $(-1)^k\binom{t+1}{k} = (-1)^k\binom{t}{k} + (-1)^k\binom{t}{k-1}$

$= (-1)^k\binom{t}{k}$ - $(-1)^{k-1}\binom{t}{k-1}$ and

$$G_n^{t+1} = \sum_{k=0}^{t+1}(-1)^k \binom{t+1}{k} T_{n+t+1-k}$$

$$= \sum_{k=0}^{t+1}((-1)^k\binom{t}{k} - (-1)^k\binom{t}{k-1})T_{n+t+1-k}$$

$$= \sum_{k=0}^{t+1}(-1)^k \binom{t}{k} T_{n+t+1-k} - \sum_{k=0}^{t+1}(-1)^k \binom{t}{k-1} T_{n+t+1-k}$$

$$= \sum_{k=0}^{t}(-1)^k \binom{t}{k} T_{n+t+1-k} - \sum_{k=1}^{t+1}(-1)^k \binom{t}{k-1} T_{n+(t+1)-k)}$$

$$= \sum_{k=0}^{t}(-1)^k \binom{t}{k} T_{n+t+1-k} - \sum_{k=1}^{t}(-1)^k \binom{t}{k} T_{n+t-k}$$

$$= G_{n+1}^t - G_n^t \qquad \text{(VII)}$$

Conclusion: We are applying for the *finite difference operator* $\Delta^{(i)}$,
$\boldsymbol{G_n^i = \left(\Delta^{(i)} T\right)_n = \sum_{k=0}^{i}(-1)^k \binom{i}{k} T_{n+i-k}}$, and from (VI) and (VII) P(t+1) is true.
⊡

Since the Tribonacci sequence T_n satisfies a linear recurrence with constant coefficients, the finite differences of order i also satisfy a linear recurrence of the same order , and in fact, under convolution with binomial coefficients, the resulting sequence G_n^i will still satisfy the same Tribonacci recurrence, though it will be a different Tribonacci sequence (with shifted initial value).■

Proposition 1: For fixed, $i \geq 1$, $\boldsymbol{G_n^i = \left(\Delta^{(i)} T\right)_n = \sum_{k=0}^{i} (-1)^k \binom{i}{k} T_{n+i-k}}$,

is a Tribonacchi sequence in n for $n \geq 3$ and that is

$\boldsymbol{G_n^i = G_{(n-1)}^i + G_{(n-2)}^i + G_{(n-3)}^i}$ **for** $n \geq 6$.

Theorem 3

In the same way as we proved (*) one can show that (**), $F_n^i = \sum_{k=0}^{i} \binom{i}{k} T_{n+i-k}$ holds true.

That is $\left\{F_n^{(i)}\right\}_{n\geq 3}$ is a Tribonacci sequence.

For illustration

Let $T_0 = 0, \quad T_1 = 1, \quad T_2 = 1$

$T_n = T_{n-3} + T_{n-2} + T_{n-1} \quad , \quad \forall \geq 3$

$$\begin{cases} G_n{}^1 = T_{n+1} + T_n \\ G_n{}^2 = G_{n+1}^1 + G_n{}^1 \\ G_n{}^3 = G_{n+1}{}^2 + G_n{}^2 \\ G_n{}^4 = G_{n+1}{}^3 + G_n{}^3 \\ \\ G_n^{i+1} = G_{n+1}^i + G_n^i, \forall\, i \geq 1 \text{ and fixed } n \geq 1 \end{cases}$$

$$G_n^i = \sum_{k=0}^{i} \binom{i}{k} T_{n+k}$$

$$G_1^1 = T_1 + T_2 G_1^2 = T_1 + 2T_2 + T_3 G_1^3 = T_1 + 3T_2 + 3T_3 + T_4$$
$$G_1^4 = T_1 + 4T_2 + 6T_3 + 4T_4 + T_5$$

$$G_2^1 = T_2 + T_3 G_2^2 = T_2 + 2T_3 + T_4 G_2^3 = T_2 + 3T_3 + 3T_4 + T_5$$
$$G_2^4 = T_2 + 4T_3 + 6T_4 + 4T_5 + T_6$$

$$G_3^1 = T_3 + T_4 G_3^2 = T_3 + 2T_4 + T_5 G_3^3 = T_3 + 3T_4 + 3T_5 + T_6$$
$$G_3^4 = T_3 + 4T_4 + 6T_5 + 4T_6 + T_7$$

$$G_4^1 = T_4 + T_5 G_4^2 = T_6 + 2T_5 + T_4 G_4^3 = T_4 + 3T_5 + 3T_6 + T_7$$
$$G_4^4 = T_4 + 4T_5 + 6T_6 + 4T_7 + T_8$$

$$G_5^1 = T_5 + T_6 G_5^2 = \ T_7 + 2T_6 + \ T_5 G_5^3 = T_5 + 3T_6 + \ 3T_7 + \ T_8$$

$$G_5^4 = T_5 + 4T_6 + \ 6T_7 + \ 4T_8 + T_9$$

$$G_6^1 = T_6 + T_7 G_6^2 = \ T_8 + 2T_7 + \ T_6 G_6^3 = T_6 + 3T_7 + \ 3T_8 + \ T_9$$
$$G_6^4 = T_6 + 4T_7 + \ 6T_8 + \ 4T_9 + T_{10}$$

$$G_n^{i+1} = G_{n+1}^i \ + \ \ G_n^i \ , \ \forall\, i \ \geq 1 \ \text{ and fixed } \ \ n \geq 1$$

$$G_1^1 = T_1 + T_2 G_1^2 = G_2^1 + G_1^1 G_1^3 = G_2^2 + \ G_1^2 G_1^4 = G_2^3 + G_1^3$$

$$G_2^1 = T_2 + T_3 G_2^2 = G_3^1 + \ G_2^1 G_2^3 = G_3^2 + G_2^2 G_2^4 = G_3^3 + G_2^3$$

$$G_3^1 = T_3 + T_4 G_3^2 = G_4^1 + G_3^1 G_3^3 = G_4^2 + \ G_3^2 G_3^4 = G_4^3 + G_3^3$$

$$G_4^1 = T_4 + T_5 G_4^2 = G_5^1 + \ G_4^1 G_4^3 = G_5^2 + G_4^2 G_4^4 = G_5^3 + G_4^3$$

$$G_5^1 = T_5 + T_6 G_5^2 = G_6^1 \ + \ G_5^1 G_5^3 = G_6^2 + \ G_5^2 G_5^4 = G_6^3 + G_5^3$$

$$G_6^1 = T_6 + T_7 G_6^2 = G_7^1 \ + \ G_6^1 G_6^3 = G_7^2 + \ G_6^2 G_6^4 = G_7^3 + G_6^3$$

Define a set S_i which is the family of sets .

$S_i \ = \{G_t^i : \ t \geq 1 \ \}$ and $S^i = \{G_i^t : \ t \geq 1 \}$.
For instance $S_1 = \{G_t^1 : \ t \geq 1\} = \{G_1^1 , \ G_2^1 , \ G_3^1 , . \ . \ . \ \}$ and $S^1 = \{G_1^t : \ t \geq 1 \} =$ $\{G_1^1 , \ G_1^2 , \ G_1^3 , . \ . \ . \}$, respectively.

Tn	$G_n{}^1 = T_{n+1} - T_n$	$G_n{}^2 = G_{n+1}{}^1 - G_n{}^1$	$G_n{}^3 = G_{n+1}{}^2 - G_n{}^2$	$G_n{}^4 = G_{n+1}{}^3 - G_n{}^3$	$G_n{}^5 = G_{n+1}{}^4 - G_n{}^4$
0	1	-1	2	-2	2
1	0	1	0	0	2
1	1	1	0	2	-2
2	1	0	2	-2	2
3	2	1	2	0	2
5	3	1	2	2	2
8	6	3	2	0	0
11	13	7	4	4	2
24	20	13	14	12	12
44	37	31	26	22	18
81	68	57	48	40	34
149	125	105	88	74	
274	230	193	162		
504	423	355			
927	778				
1705					

$$G_n^1 G_n^2 G_n^1 + G_n^2 G_n^3 G_n^1 + \ G_n^2 + G_n^3 G_n^4 G_n^3 \ + G_n^4$$

0	1	**1**	3	**4**	8	**11**
1	2	**3**	5	**8**	14	**19**
1	3	**4**	9	**13**	26	**35**
2	6	**8**	17	**25**	48	**65**
4	11	**15**	31	**46**	88	**119**
7	20	**27**	57	**84**	162	**219**
13	37	**50**	105	**155**	298	**403**
24	68	**92**	193	**285**	548	**741**
44	125	**169**	355	**524**	1008	**1363**
81	230	**321**	653	**974**	1854	**2507**
149	423	**572**	1201	**1773**	3410	**4611**
274	778	**1052**	2209	**3261**	6272	**8481**
504	1431	**1935**	4063	**5998**	11536	**15599**
927	2632	**3559**	7473	**11032**	21218	**28691**
1705	4841	**6546**	13745	**20191**	39026	**52771**

A	**B**	**A+B**	**C**	**A+B+C**	**D**	**A+B+C+D**
$T_n = T_{n-3} + T_{n-2} + T_{n-1}$	$G_n^1 = T_{n+1} + T_n$		$G_n^2 = G_{n+1}^1 + G_n^1$		$G_n^3 = G_{n+1}^2 + G_n^2$	
0	1	1	3	4	8	12
1	2	3	5	8	14	22
1	3	4	9	13	26	39
2	6	8	17	25	48	73
4	11	15	31	46	88	134
7	20	27	57	84	162	246
13	37	50	105	155	298	453
24	68	92	193	285	548	833
44	125	169	355	524	1008	1532
81	230	321	653	964	1854	2818
149	423	572	1201	1773	3410	4183

Proposition 2: The sequence T_n is an odd function.

We Redefine the Tribonacci sequence so that it behaves like an odd function., that is $T_{-n} = -T_n$.
And we are keeping only the Initial condition: $T_0 = 0$.

Remark: We define $T_{-n} = -T_n$, $\forall n$, then it is just saying that:

- T_n is odd-symmetric about 0, like $f(-x) = -f(x)$

So, for any n, if the property holds for n, it will hold true for $(n+1)$, because

$$\boldsymbol{T}_{-(n+1)} = -\boldsymbol{T}_{(n+1)}$$

Theorem4:

We define the Tribonacchi sequence so that $T_{-n} = -T_n$ and start with $T_0 = 0$, then the property
$\boldsymbol{T}_{-(n+1)} = -\boldsymbol{T}_{(n+1)}$follows naturally for all$\forall \boldsymbol{n} \geq \boldsymbol{0}$ **.**

Proof

We prove by induction that $T_{-n} = -T_n$, $\forall \boldsymbol{n} \geq \boldsymbol{0}$ **.**
For n = 0. Then $T_{-0} = T_0 = 0 = -T_0$.
Assume that:
$T_{-k} = -T_k \forall k \leq n$ **.**
We want to show that
$\boldsymbol{T}_{-(n+1)} = -\boldsymbol{T}_{(n+1)}$ **.**
Using the Tribonacchi recurrence formula also the reversed one:
The reverse recurrence formula:

$$\boldsymbol{T}_{-(n+1)} = \boldsymbol{T}_{-(n+4)} = \boldsymbol{T}_{-(n+1)+3} - \boldsymbol{T}_{-(n+1)+2} - \boldsymbol{T}_{-(n+1)+1}$$
$$= \boldsymbol{T}_{-(n-2)} - \boldsymbol{T}_{-(n-1)} - \boldsymbol{T}_{-n}$$

Now applying the induction hypothesis:
$\boldsymbol{T}_{-(n-2)} = -\boldsymbol{T}_{(n-2)}$, $\boldsymbol{T}_{-(n-1)} = -\boldsymbol{T}_{-(n-1)}$ **and** $\boldsymbol{T}_{-n} = -\boldsymbol{T}_{n}$

So, plugging in we have
$\boldsymbol{T}_{-(n+1)} = -\boldsymbol{T}_{(n+1)}$ induction step holds.

Conclusion: By mathematical induction, we shown: $\boldsymbol{T}_{-n} = -\boldsymbol{T}_{n} \forall \boldsymbol{n} \geq \boldsymbol{0}$

Provided we define the following Tribonacchi sequence with

- $\boldsymbol{T}_0 = \boldsymbol{0}$
- Reversed recurrence: $\boldsymbol{T}_{n-3} = \boldsymbol{T}_n - \boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-2}$

And Symmetry enforced as a part of a rule

Theorem 5

For set A $= \boldsymbol{T}_{-n} \cup \{0\} \cup \boldsymbol{T}_n$, $(A, +)$ is an infinite abelian group.

Proof:

$$\{-,-,-,-7,-4,-2,-1,-1,0,1,1,2,4,7,-,-,-\}$$

$$= \{-,-,-,-7,-4,-2,-1,-1\} \cup \{0\} \cup \{1,1,2,4,7,-,-,-\}$$

$$= \boldsymbol{T}_{-n} \cup \{0\} \cup \boldsymbol{T}_n = A$$

A is the of Symmetric Inversed sequence. The identity is $= 0 \in A$. For each element $=$ $x \in A$, there exist $u = -x \in A$ such that $m + u = 0$. This shows every element is invertible.

Moreover, $x + y = y + x$ because formal addition property, and A is an infinite set. Hence $(A , +)$ is infinite abelian group.

Theorems 6*:* Consider the sets

I. $G_n = T_{n+1} - T_n$, and $F_n = T_{n+1} + T_n$. $\forall\, n \geq 3$.

$(G_n , +)$ and $(F_n , +)$ are Infinite abelian groups.

2. $(G_n^i , +)$ and $(F_n^i , +)$ where

$$G_n^i = \sum_{k=0}^{i} (-1)^k \binom{i}{k} T_{n+i-k} \quad \text{and} \quad F_n^i = \sum_{k=0}^{i} \binom{i}{k} T_{n+i-k}$$

are infinite abelian groups.

Proof. It follows from the fact that both sets are Tribonacci Sequences. In particular they are symmetric Inversed sequence, $G_{-n} = -G_n$, $F_{-n} = -F_n$, $G_{-n}^i = -G_n^i$, $F_{-n}^i = -F_n^i$ and the operation is closed. (Theorem 1- 6). ■

Example. See the tables above.
Given a two-by-Two matrix

$$X = \begin{pmatrix} T_{n+1} & T_n \\ T_n & T_{n+1} \end{pmatrix}$$

Where $T = \{x : x = T_n , \ for\, n = 0, 1 , 2 ,3 , - , - , -\}$
$= \{0 ,1, 1, 2, 4, 7, 13, 24, - - -\}$
$= \{ T_0, T_1 , T_2 , T_3 , T_4 , T_5 , T_6 , T_7 , - - - \}$

Consider $S_n = \left\{X: X = \begin{pmatrix} T_{n+1} & T_n \\ T_n & T_{n+1} \end{pmatrix}, \ n \in \mathbb{Z}^+ , T_n \in T \right\}$

Theorem 7:
For each $X \in S_n$,

$$X^m = \begin{pmatrix} T_{n+1} & T_n \\ T_n & T_{n+1} \end{pmatrix}^m = \begin{pmatrix} \dfrac{(F_n)^m + (G_n)^m}{2} & \dfrac{(F_n)^m - (G_n)^m}{2} \\ \dfrac{(F_n)^m - (G_n)^m}{2} & \dfrac{(F_n)^m + (G_n)^m}{2} \end{pmatrix}$$

Proof:

Set $T_{n+1} = a$ and $T_n = b$ so that $\begin{pmatrix} T_{n+1} & T_n \\ T_n & T_{n+1} \end{pmatrix}^m = \begin{pmatrix} a & b \\ b & a \end{pmatrix}^m$

We can diagonalizable X because the matrix is symmetric.
Consider $(X - \lambda I) = \begin{pmatrix} a & b \\ b & a \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix}$

To find the Eigenvalue, we are solving the equation

$$det(X - \lambda I) = det \begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} = 0$$

This gives, $(a-\lambda)^2 - b^2 = 0$, and solving for λ,
$\lambda = (a+b)$ and $\lambda = (a-b)$.

Next we find the Eigen vector(s);
Case I. When $\lambda = (a+b)$, we have

$$\begin{pmatrix} (a+b)-(a+b) & b \\ b & (a+b)-(a-b) \end{pmatrix} = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$$

When solving the equation $bx_1 + 0x_2 = 0$, we have Eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Case II. When $\lambda = (a-b)$, we have

We have

$$\begin{pmatrix} (a+b)-(a-b) & b \\ b & (a+b)-(a-b) \end{pmatrix} = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$$

$\Longrightarrow$ The solution gives the Eigenvector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The next step will be diagonalization. Let $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$

As $X = P D P^{-1}$ we have $X^m = (PDP^{-1})^m = P D^m P^{-1}$.

$D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} \Rightarrow D^m = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}^m = \begin{bmatrix} (a+b)^m & 0 \\ 0 & (a-b)^m \end{bmatrix}$

$\Longrightarrow X^m = (PDP^{-1})^m = P D^m P^{-1} = P \begin{bmatrix} (a+b)^m & 0 \\ 0 & (a-b)^m \end{bmatrix} P^{-1}$

For $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we have $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

This implies, $X^m = (PDP^{-1})^m = P D^m P^{-1} = P \begin{bmatrix} (a+b)^m & 0 \\ 0 & (a-b)^m \end{bmatrix} P^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} (a+b)^m & 0 \\ 0 & (a-b)^m \end{bmatrix} * \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Commitment

$$= \frac{1}{2} * \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} (a+b)^m & 0 \\ 0 & (a-b)^m \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} * \begin{bmatrix} (a+b)^m + (a-b)^m & (a+b)^m - (a-b)^m \\ (a+b)^m - (a-b)^m & (a+b)^m + (a-b)^m \end{bmatrix}$$

$$\Rightarrow X^m = \begin{bmatrix} \frac{(a+b)^m + (a-b)^m}{2} & \frac{(a+b)^m - (a-b)^m}{2} \\ \frac{(a+b)^m - (a-b)^m}{2} & \frac{(a+b)^m + (a-b)^m}{2} \end{bmatrix}$$

$$= \begin{pmatrix} \frac{(F_n)^m + (G_n)^m}{2} & \frac{(F_n)^m - (G_n)^m}{2} \\ \frac{(F_n)^m - (G_n)^m}{2} & \frac{(F_n)^m + (G_n)^m}{2} \end{pmatrix} \qquad \blacksquare$$

Theorem 8:
Let $R = \left\{ a * \begin{pmatrix} A & B \\ B & A \end{pmatrix} : \; A = \frac{(F_n)^m + (G_n)^m}{2}, \text{and } B = \frac{(F_n)^m - (G_n)^m}{2}, \; a \in (2\mathbb{Z}) \right\}$

$(R, +)$, is an abelian group of infinite set.

Proof:
Let $M = a_1 * \begin{pmatrix} A_1 & B_1 \\ B_1 & A_1 \end{pmatrix}$ and $N = a_2 * \begin{pmatrix} A_2 & B_2 \\ B_2 & A_2 \end{pmatrix} a_1, a_2 \in \mathbb{Z}$.

Then $M - N = a_1 * \begin{pmatrix} A_1 & B_1 \\ B_1 & A_1 \end{pmatrix} + a_2 * \begin{pmatrix} A_2 & B_2 \\ B_2 & A_2 \end{pmatrix}$

$$= (a_1 - a_2) * \left(\begin{pmatrix} A_1 & B_1 \\ B_1 & A_1 \end{pmatrix} - \begin{pmatrix} A_2 & B_2 \\ B_2 & A_2 \end{pmatrix} \right)$$

$$= (a_1 - a_2) * \left(\begin{pmatrix} A_1 - A_2 & B_1 - B_2 \\ B_1 - B_2 & A_1 - A_2 \end{pmatrix} \right)$$

$$= a_3 * \begin{pmatrix} A_1 - A_2 & B_1 - B_2 \\ B_1 - B_2 & A_1 - A_2 \end{pmatrix} \text{ where } \; a_3 = (a_1 - a_2) \in (2\mathbb{Z})$$

$$= a_3 * \begin{pmatrix} (F_{n_1})^m + \left(G_{n_1}\right)^m - (F_{n_2})^m - (G_{n_2})^m & (F_{n_1})^m - \left(G_{n_1}\right)^m - (F_{n_2})^m + (G_{n_2})^m \\ (F_{n_1})^m - \left(G_{n_1}\right)^m - (F_{n_2})^m + (G_{n_2})^m & (F_{n_1})^m + \left(G_{n_1}\right)^m - (F_{n_2})^m - (G_{n_2})^m \end{pmatrix}$$

$$= a_3 * \begin{pmatrix} (F_{n_1})^m - (F_{n_2})^m + \left(G_{n_1}\right)^m - (G_{n_2})^m & (F_{n_1})^m - (F_{n_2})^m + (G_{n_2})^m - (G_{n_1})^m \\ (F_{n_1})^m - (F_{n_2})^m + (G_{n_2})^m - (G_{n_1})^m & (F_{n_1})^m - (F_{n_2})^m + \left(G_{n_1}\right)^m - (G_{n_2})^m \end{pmatrix}$$

$(F_{n_1})^m + \left(G_{n_1}\right)^m - (F_{n_2})^m - (G_{n_2})^m$ and $(F_{n_1})^m - (F_{n_2})^m + \left(G_{n_1}\right)^m - (G_{n_2})^m$

Are sum of two Tribonacchi like sequences which is Tribonacchi sequence. Thus $(M - N) \in R$ and follows R is an abelian group.

■

Conclusion:

Further research is being conducted on the generalized Tribonacchi sequence, specifically the sequence $S_1, S_2, S_3, \ldots, S_n$, where

$$S_n = T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-1}S_3$$

The goal is to extend this investigation to explore the underlying infinite group structures related to matrix multiplication and function composition. These findings offer compelling examples that bridge number theory with group theory, highlighting rich mathematical interconnection.

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