e-ISSN 3050-9645
p-ISSN 2795-3274

Check for updates

GPH International Journal of Mathematics
WWW.GPHJOURNAL.ORG

10.5281/zenodo.16927158 Vol. 8 Issue 07 July - 2025 Manuscript ID: #2024

Positive solutions for a coupled system of non linear of second-order boundary-valued differential equations

By:

Miguel A. Yglesias Jauregui
Instituição: Universidad Nacional Autónoma
de Tayacaja Daniel Hernandez Morillo
Jr. San Martin 459 -Pampas-Tayacaja, Huancavelica-Perú.
E-mail: miguelyglesias@unat.edu.pe

Milton M. Cortez Gutiérrez
Instituição: Universidad Nacional de Trujillo
Av Juan Pablo II s/n , Ciudad Universitária. Trujillo. La Libertad-Perú.
E-mail: mcortez@unitru.edu.pe

Hernan O. Cortez Gutierrez
Universidad Nacional del Callao-Callao-Peru
hocortez@unacvirtual.edu.pe
Corresponding author: mcortez@unitru.edu.pe

Abstract

The main objective of this article is concerned with existence and uniqueness of the positive solutions for a coupled system of second-order boundary-valued differential equations, which the boundary conditions are coupled by integrals. For that one, the solutions are related to Green's functions and represented by integral equations and so that, we used a cone compression and expansion fixed point theorem applied to a completely continuous operator, as seen in Guo, D. and Lakshmikant V (1988).

Keywords:

Positive solution, system of differential equations, Green functions, fixed point theorem.

How to cite: Jauregui, M. A., Gutiérrez, M. M., & Gutierrez, H. O. (2025). Positive solutions for a coupled system of non linear of second-order boundary-valued differential equations. *GPH-International Journal of Mathematics*, 8(7), 157-175.
https://doi.org/10.5281/zenodo.16927158

1. Introduction.

This research analyzes a system of nonlinear ordinary differential equations, with regard to the existence of positive solutions and uniqueness of solutions. Evidently, different methods can be developed from this perspective, which can provide a deterministic solution or perform a simulation that demonstrates its geometric behavior or apply approximation techniques. One of the problems studied, from which several results have been obtained, is the system of two second-order ordinary differential equations. Several researchers, such as Asif et al. (2010) and Asif and Khan (2012), have demonstrated the existence of positive solutions, depending on the characteristics imposed on the boundary conditions. In this sense, this paper consists of considering the system of second-order differential equations in the form:

$$\begin{cases} x''(t) + a_1(t)x'(t) + b_1(t)x(t) + f_1\big(t, x(t), y(t)\big) = 0 \ \ , t \in (0,1) \\ y''(t) + a_2(t)y'(t) + b_2(t)y(t) + f_2\big(t, x(t), y(t)\big) = 0 \ , t \in (0,1) \end{cases} \qquad (1)$$

With the boundary conditions

$$x(0) = \int_0^1 y(t)d\alpha(t) \ \ , y(0) = \int_0^1 x(t)d\beta(t) \qquad (2)$$

$$x(1) = 0 \ \ , \ \ y(1) = 0 \qquad (3)$$

Where α y β are continuous on the right on $[0,1\rangle$, left continuous at$t = 1$, and non decreasing on [0,1], with $\alpha(0)$=$\beta(0)$=0, where $\int u(s)d\alpha(s)$ and

$\int (s)\beta(s)$ denote Riemann Stieltjes's integrals of u with respect to α and β. Beyond that, we have

$f \in C\ ((0,1) \times [0,+\infty) \times (0,+\infty), [0,+\infty))$,

$g \in C\big((0,1) \times (0,+\infty) \times [0,+\infty), [0,+\infty)\big)$ For the case in two variables one can have the system given in **Cortez, M., Quique, J. (2023).** We can rewrite (1),(2),(3) like a system of first order differential equations such as in **Cassius Vinícius Casaro, Diego Márcio Gonzaga, Marco Rogério da Silva Richetto Kátia, Celina da Silva Richetto.(2022)**

2. Materials and methods

Let E be a Banach space, a non-empty convex subset K of E is a cone, if satisfies the following conditions

1). For $\in K$, $\lambda \geq 0$, then $\lambda x \in K$

2). If $x \in K$, $-x \in K$, then$x = 0$.

Let E and F be normed spaces over the field $\mathbb{K}$, If$T: E \longrightarrow F$is a linear operator. Then we have (a) T is compact if, and only if, for every bounded set $A \subset E$ it follows that (A) is relatively compact.

(b) If $(\overline{B}(0,1))$ is relatively compact, then T is compact.

(c) If T is compact, then T is bounded.

Compression and expansion fixed point theorem (see Guo, D. and LakshmikantV (1988).)

Let K be a cone and Ω_1, Ω_2be two open and bounded subsets of a Banach space E, such that $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$and $T: K \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow K$ a completely continuous operator. Suppose that one of the conditions holds

C1) $\|Tu\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_1and\|Tu\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_2$

or

C2) $\|Tu\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_1and\|Tu\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_2$

Then T has at least one fixed point in$K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.We reacall that a distribution over (0,1), is a linear and continuous functional in $C_0^\infty(0,1)$ (infinitely differentiable functions with compact support in (0,1)) and represented by$\langle T, \varphi \rangle$.We denote the set of distributions by $\mathcal{D}'(0,1)$. A fundamental solution of an ordinary differential equation $Lu = f$is when it can be obtained by solving $Lu = \delta$. Where δ is Dirac's distribution. This article focuses the systems of ordinary differential equations mentioned above (1)-(3), which will be rewritten by changing the variable.

$$u(t) = e^{\frac{1}{2}\int_0^t a_1(t)dt}x(t) \quad , \quad v(t) = e^{\frac{1}{2}\int_0^t a_2(t)dt}y(t)$$

Choosing the coefficients appropriately $a_1(t), b_1(t), a_2(t)$y$b_2(t)$, it yields

$$\begin{cases} -u''(t) = f(t, u(t), v(t)) \\ -v''(t) = g(t, u(t), v(t)) \end{cases} \qquad (4)$$

$$u(1) = 0, \qquad v(1) = 0 \quad , \qquad (5)$$

$$u(0) = \int_0^1 \sigma_1(t)\, v(t) d\alpha(t)\, , v(0) = \int_0^1 \sigma_2(t)\, u(t) d\beta(t) \qquad (6)$$

Where , $\sigma_1(t) = e^{-\frac{1}{2}\int_0^t a_1(t)dt}$, $\sigma_2(t) = e^{-\frac{1}{2}\int_0^t a_2(t)dt}$. We asume that $\sigma_1, \sigma_2 \in L^\infty(0,1)$

3. The main result

Before proving the main result, we consider some hypotheses

H1) For $f \in C^0\big((0,1) \times [0,+\infty) \times (0,+\infty), [0,+\infty)\big)$ it holds that $f(t,u,v)$ is increasing with respect to u and decreasing with respect to v, moreover, there exists $\lambda_1, \mu_1 \in [0,1)$ such that $\forall u, v > 0$ and $a \in (0,1)$ one has

$$a^{\lambda_1} f(t,u,v) \leq f(t,au,v) \qquad (7)$$

$$f(t,u,av) \leq a^{-\mu_1} f(t,u,v) \qquad (8)$$

$$0 < \int_0^1 f(t,1,1-t)dt < +\infty \qquad (9)$$

Consequence 1: For $a > 1$ we have

$f(t,au,v) \leq a^{\lambda_1} f(t,u,v)$, $\forall u, v > 0$, $f(t,u,v) \leq a^{\mu_1} f(t,u,av)$, $\forall u, v > 0$. In fact, $1/a \in (0,1)$ consecuently de (7) se tiene:

$$(1/a)^{\lambda_1} f(t,u,v) \leq f\left(t, \frac{1}{a}u, v\right) \Rightarrow f(t,u,v) \leq a^{\lambda_1} f\left(t, \frac{1}{a}u, v\right) \Rightarrow f(t,au,v) \leq a^{\lambda_1} f(t,u,v)$$

In the same way, $f\left(t,u,\frac{v}{a}\right) \leq (1/a)^{-\mu_1} f(t,u,v) \Rightarrow f(t,u,v) \leq a^{\mu_1} f(t,u,av)$

Consequence 2: $0 < \int_0^1 f(t,1-t,1)dt < +\infty$.In fact, by using(7), letting $u = 1-t$, $v = 1$, we obtain $f(t,1-t,1) \leq a^{-\lambda_1} f(t,a(1-t),1) \qquad (i)$

On the other hand, by the relation (8), letting $u = a(1-t)$, $av = 1$, it yields

$f(t,a(1-t),1) \leq a^{-\mu_1} f\left(t, a(1-t), \frac{1}{a}\right)$. Since $a(1-t) < 1 \Rightarrow \frac{1}{a} > (1-t)$,

f is increcreasing in the second variable and decreasing in the third variable, we obtain

$$f(t,a(1-t),1) \leq a^{-\mu_1} f\left(t,a(1-t),\frac{1}{a}\right) \leq a^{-\mu_1} f(t,1,1-t) \qquad (ii)$$

So that, from (i) and (ii) $f(t,1-t,1) \leq a^{-\lambda_1} f(t,a(1-t),1) \leq a^{-\lambda_1} a^{-\mu_1} f(t,1,1-t)$

Consequently, by integrating from 0 to 1 and taking into account (6) we have

$$0 < \int_0^1 f(t,1-t,1)dt \leq a^{-(\lambda_1+\mu_1)} \int_0^1 f(t,1,1-t)dt < +\infty$$

Analogously, for $g \in C^0\big((0,1) \times (0,+\infty) \times [0,+\infty), [0,+\infty\big)$it verifies

$g(t,u,v)$is decreasing with respect to u and increasing in v, moreover, there exists $\lambda_2, \mu_2 \in [0,1)$ such that $\forall u, v > 0$ and $a \in (0,1)$ it yields

$$a^{\lambda_2} g(t,u,v) \leq g(t,u,av) \qquad (10)$$

$$g(t,au,v) \leq a^{-\mu_2} g(t,u,v) \qquad (11)$$

$$0 < \int_0^1 g(t,1-t,1)dt < +\infty \qquad (12)$$

<u>**Consequence 3:**</u> For $a > 1$we have $g(t,u,av) \leq a^{\lambda_2} g(t,u,v)$ $,\forall u, v > 0$

In fact, $1/a \in (0,1)$ consequently de (10)

$$(1/a)^{\lambda_2} g(t,u,v) \leq g\left(t,u,\frac{v}{a}\right) \Rightarrow g(t,u,v) \leq a^{\lambda_2} g\left(t,u,\frac{v}{a}\right) \Rightarrow g(t,u,av)$$
$$\leq a^{\lambda_2} g(t,u,v)$$

Applying (11).We have $g\left(t,\frac{u}{a},v\right) \leq (1/a)^{-\mu_2} g(t,u,v) \Rightarrow$

$$g(t,u,v) \leq a^{\mu_2} g(t,au,v) \Rightarrow 0 < \int_0^1 g(t,1,1-t)dt < +\infty$$

H2) If $\xi_1 = \int_0^1 (1-t)\sigma_1(t)\, d\alpha(t)$, $\xi_2 = \int_0^1 (1-t)\sigma_2(t)\, d\beta(t)$, we have $\xi_1 > 0$, $\xi_2 > 0$, and $\xi = 1 - \xi_1 . \xi_2 > 0$. We know that $-u''(t) = \delta(t-s)$ with $u(0) = 0$, $u(1) = 0$

Has a solution $u(t)$ as being the Green function $G(t,s)$

$$G(t,s) = \begin{cases} (1-s)t \ , & 0 \le t < s < 1 \\ s(1-t) \ , & 0 < s < t \le 1 \end{cases}$$

We apply Duhamel´s principle to each differential equation of the system (4)-(6):

$$-u''(t) = f(t, u(t), v(t)), u(0) = 0 \ , \ u(1) = 0 \ , \text{ so that}$$

$u(t) = \int_0^1 G(t,s)\, f(s, u(s), v(s)) ds$. On the other hand

$$-u''(t) = 0, u(0) = \int_0^1 \sigma_1(t)\, v(t) d\alpha(t) \ , u(1) = 0, u(t) = (1-t)u(0), \text{ therefore}$$

$$u(t) = (1-t)u(0) + \int_0^1 G(t,s)\, f(s, u(s), v(s)) ds \qquad (13)$$

$$v(t) = (1-t)v(0) + \int_0^1 G(t,s)\, g(s, u(s), v(s)) ds \qquad (14)$$

By simplicity of notation. Letting, $p(s) = f\big(s, u(s), v(s)\big)$, $q(s) = g\big(s, u(s), v(s)\big)$. Now, we find an integral representation of the differential system (4)-(6), Multiplying the functions obtained in the relationships (13),(14) by $\sigma_2(t)$ and $\sigma_1(t)$ respectively, by integrating with respect to $d\beta(t)$ and $d\alpha(t)$ from 0 to 1. We have

$$\int_0^1 \sigma_2(t)\, u(t) d\beta(t) = u(0) \int_0^1 (1-t)\, \sigma_2(t) d\beta(t) + \int_0^1 \sigma_2(t) \left(\int_0^1 G(t,s)\, p(s) ds \right) d\beta(t)$$

$\int_0^1 \sigma_1(t)\, v(t) d\alpha(t) = u(0) \int_0^1 (1-t)\, \sigma_2(t) d\alpha(t) + \int_0^1 \sigma_2(t) \left(\int_0^1 G(t,s)\, p(s) ds \right) d\alpha(t)$. I ndeed

$$v(0) - u(0) \int_0^1 (1-t)\, \sigma_2(t) d\beta(t) = \int_0^1 \sigma_2(t) \left(\int_0^1 G(t,s)\, p(s) ds \right) d\beta(t)$$

$u(0) - v(0) \int_0^1 (1-t)\, \sigma_1(t) d\alpha(t) = \int_0^1 \sigma_1(t) \left(\int_0^1 G(t,s)\, q(s) ds \right) d\alpha(t)$. Hence

$$\begin{pmatrix} -\xi_2 & 1 \\ 1 & -\xi_1 \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} \int_0^1 \sigma_2(t) \left(\int_0^1 G(t,s)\, p(s) ds \right) d\beta(t) \\ \int_0^1 \sigma_1(t) \left(\int_0^1 G(t,s)\, q(s) ds \right) d\alpha(t) \end{pmatrix}, \text{ where}$$

$\xi_1 = \int_0^1 (1-t)\, \sigma_1(t) d\alpha(t)$, $\xi_2 = \int_0^1 (1-t)\, \sigma_2(t) d\beta(t)$, with determinant $\xi = 1 - \xi_1 \xi_2 \neq 0$

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \frac{\begin{pmatrix} \xi_1 & 1 \\ 1 & \xi_2 \end{pmatrix}}{\xi} \begin{pmatrix} \int_0^1 \sigma_2(t)\left(\int_0^1 G(t,s)\,p(s)ds\right)d\beta(t) \\ \int_0^1 \sigma_1(t)\left(\int_0^1 G(t,s)\,q(s)ds\right)d\alpha(t) \end{pmatrix}$$

$$u(0) = \frac{\xi_1}{\xi}\int_0^1 \sigma_2(t)\left(\int_0^1 G(t,s)\,p(s)ds\right)d\beta(t) + \frac{1}{\xi}\int_0^1 \sigma_1(t)\left(\int_0^1 G(t,s)\,q(s)ds\right)d\alpha(t) \quad (15)$$

$$v(0) = \frac{1}{\xi}\int_0^1 \sigma_2(t)\left(\int_0^1 G(t,s)\,p(s)ds\right)d\beta(t) + \frac{\xi_2}{\xi}\int_0^1 \sigma_1(t)\left(\int_0^1 G(t,s)\,q(s)ds\right)d\alpha(t) \quad (16)$$

Applying Fubini's Theorem to (15) ,(16) and substituting to the equations (13) and(14), it yields

$$u(t) = \frac{(1-t)\xi_1}{\xi}\int_0^1 p(s)\left(\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t)\right)ds + \frac{(1-t)}{\xi}\int_0^1 q(s)\left(\int_0^1 \sigma_1(t)G(t,s)\,d\alpha(t)\right)ds + \int_0^1 G(t,s)\,p(s)ds$$

$$u(t) = \int_0^1 U_1(t,s)\,p(s)ds + \int_0^1 U_2(t,s)\,q(s)ds \quad (17)$$

Where

$$U_1(t,s) = \frac{(1-t)\xi_1}{\xi}\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t) + G(t,s), U_2(t,s) = \frac{(1-t)}{\xi}\int_0^1 \sigma_1(t)G(t,s)\,d\alpha(t)$$

$$v(t) = \frac{(1-t)}{\xi}\int_0^1 p(s)\left(\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t)\right)ds + \frac{(1-t)\xi_2}{\xi}\int_0^1 q(s)\left(\int_0^1 \sigma_1(t)G(t,s)\,d\alpha(t)\right)ds +$$

$$+\int_0^1 G(t,s)\,q(s)ds$$

$$v(t) = \int_0^1 V_1(t,s)\,p(s)ds + \int_0^1 V_2(t,s)\,q(s)ds \quad (18)$$

$V_1(t,s) = \frac{(1-t)}{\xi}\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t), V_2(t,s) = \frac{(1-t)\xi_2}{\xi}\int_0^1 \sigma_1(t)G(t,s)\,d\alpha(t) + G(t,s)$.Let$E =$

$C([0,1])$be the vector space of continuous functions on$[0,1]$with the norm$\|u\| = \max_{0\le t\le 1}|u(t)|$, which is a Banach space. Consider the set

$$K = \{(u,v) \in E \times E \;\; ; \;\; u(t), v(t) \ge \eta(1-t)\|(u,v)\|_1, t \in [0,1]\}$$

Where $\|(u,v)\|_1 = máx\{\|u\|, \|v\|\}\,, \eta = \frac{\nu}{\rho} \in [0,1]$.

$$\nu = \min\left\{\frac{\xi_1}{\xi}\int_0^1 t(1-t)\sigma_2(t)\,d\beta(t), \frac{\xi_2}{\xi}\int_0^1 t(1-t)\sigma_1(t)\,d\alpha(t), \frac{1}{\xi}\int_0^1 t(1-t)\sigma_2(t)\,d\beta(t),\right.$$
$$\left., \frac{1}{\xi}\int_0^1 t(1-t)\sigma_1(t)\,d\alpha(t)\right\}$$

$$\rho = \text{máx}\left\{\frac{\xi_1}{\xi}\int_0^1 \sigma_2(t)\,d\beta(t)+1, \frac{\xi_2}{\xi}\int_0^1 \sigma_1(t)\,d\alpha(t)+1, \frac{1}{\xi}\int_0^1 \sigma_2(t)\,d\beta(t), \frac{1}{\xi}\int_0^1 \sigma_1(t)\,d\alpha(t)\right\}$$

$K$is a non-empty closed convex set of$E \times E$. Let$K_* = K\setminus\{0\}$,$\mathcal{C}_+([0,1]) \subset E$ The space of non-negative continuous functions on$[0,1]$.We know that Green's functions satisfies

$$G(t,s). G(t,s) = t(1-t)s(1-s) \le G(t,s) \le t(1-t)(\text{o} \quad s(1-s)\ ) \quad (19)$$

We have some estimates

E1) $\begin{cases} U_i(t,s) \le \rho s(1-s)\ \ (\text{o}\ \ t(1-t)\,) \\ V_i(t,s) \le \rho s(1-s)\ \ (\text{o}\ \ t(1-t)\,) \end{cases}, \ i = 1,2 \qquad (20)$

E2) $\begin{cases} U_i(t,s) \ge \nu(1-t)s(1-s) \\ V_i(t,s) \ge \nu(1-t)s(1-s) \end{cases}, \ i = 1,2 \qquad (21)$

E3) $\begin{cases} f(t,u(t),v(t)) \le a^{\lambda_1}b^{\mu_1} f(t,1,1-t) \\ g(t,u(t),v(t)) \le a^{\lambda_2}b^{\mu_2} g(t,1-t,1) \end{cases} \qquad (22)$

E4) $\begin{cases} U_i(t,s) \ge \eta(1-t)U_i(\tau,s) \\ V_i(t,s) \ge \eta(1-t)V_i(\tau,s) \end{cases}, \ i = 1,2 \qquad (23)$

E5) $\begin{cases} U_i(t,s) \ge \eta(1-t)V_{i-(-1)^i}(\tau,s) \\ V_i(t,s) \ge \eta(1-t)U_{i-(-1)^i}(\tau,s) \end{cases}, i = 1,2 \qquad (24)$

In fact, for **E1**, by using the inequality (19) in $U_1(t,s)$

$$U_1(t,s) = \frac{(1-t)\xi_1}{\xi}\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t) + G(t,s) \le$$

$$\le \frac{\xi_1}{\xi}\int_0^1 \sigma_2(t)s(1-s)\,d\beta(t) + s(1-s) = s(1-s)\left[\frac{\xi_1}{\xi}\int_0^1 \sigma_2(t)\,d\beta(t) + 1\right] \le s(1-s)\rho$$

$$U_1(t,s) \le s(1-s)\rho$$

$$U_1(t,s) = \frac{(1-t)\xi_1}{\xi}\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t) + G(t,s) \le \frac{(1-t)\xi_1}{\xi}\int_0^1 \sigma_2(t)t(1-t)\,d\beta(t) + t(1-t)$$

Since$t \le 1$ y $1-t \le 1$, we have$U_1(t,s) \le \frac{(1-t)\xi_1}{\xi}\int_0^1 \sigma_2(t)\,d\beta(t) + (1-t) < (1-t)\rho \ \Rightarrow$

$U_1(t,s) \leq (1-t)\rho$. In the same way for$V_i(t,s)$. For**E2**, by using the inequality (19) en$U_1(t,s)$

$$U_1(t,s) = \frac{(1-t)\xi_1}{\xi}\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t) + G(t,s) \geq$$

$$\geq \frac{(1-t)\xi_1}{\xi}\int_0^1 \sigma_2(t)t(1-t)s(1-s)\,d\beta(t) + t(1-t)s(1-s) =$$

$$= \frac{(1-t)s(1-s)\xi_1}{\xi}\int_0^1 \sigma_2(t)t(1-t)\,d\beta(t) + t(1-t) \geq$$

$$\geq \frac{(1-t)s(1-s)\xi_1}{\xi}\int_0^1 \sigma_2(t)t(1-t)\,d\beta(t) \geq (1-t)s(1-s)\nu \Rightarrow U_1(t,s) \geq (1-t)s(1-s)\nu$$

With the same arguments, one verifies the other estimates of **E2)**.

For**E3)** Let$(u,v) \in K_*$,$a > 1$such that$\|(u,v)\|_1 < a \;\; \Rightarrow \;\; u(t), v(t) \geq \eta(1-t)\|(u,v)\|_1$

From consequence 1, and since f is increasing in the second variable and decreasing in the third variable, we have

$$f(t,u(t),v(t)) \leq f(t,a,\eta(1-t)\|(u,v)\|_1) \leq a^{\lambda_1} f(t,1,\eta(1-t)\|(u,v)\|_1)$$

Once again, because f is decreasing in the third variable

$$a^{\lambda_1} f(t,1,\eta(1-t)\|(u,v)\|_1) \leq a^{\lambda_1} f(t,1,\frac{\eta(1-t)\|(u,v)\|_1}{a}) \Rightarrow$$

$$f(t,u(t),v(t)) \leq a^{\lambda_1} f\left(t,1,\frac{\eta(1-t)\|(u,v)\|_1}{a}\right) = a^{\lambda_1} f\left(t,1,\frac{(1-t)}{b}\right)$$

Where $b = a/\eta\|(u,v)\|_1$, from consequence 1, $(t,u,v) \leq b^{\mu_1} f(t,u,bv)$, letting

$v = \frac{(1-t)}{b}$in the inequality, we obtain

$f(t,u(t),v(t)) \leq a^{\lambda_1} f\left(t,1,\frac{(1-t)}{b}\right) \leq a^{\lambda_1} b^{\mu_1} f(t,1,1-t)$. Analogously, it holds

$$g(t,u(t),v(t)) \leq a^{\lambda_2} g\left(t,\frac{(1-t)}{b},1\right) \leq a^{\lambda_2} b^{\mu_2} g(t,1-t,1)$$

For **E4) and E5)**. We use the estimate **E2)** and **E1)** respectively for obtaining

$$U_i(t,s) \geq \nu(1-t)s(1-s) = \rho\eta(1-t)s(1-s) \geq \eta(1-t)U_i(t,s)$$

$$U_i(t,s) \geq \nu(1-t)s(1-s) = \rho\eta(1-t)s(1-s) \geq \eta(1-t)V_{i-(-1)^i}(t,s)$$

Let $T: K_* \longrightarrow K$be the operator defined by$T(u,v) = \big(T_1(u,v), T_2(u,v)\big)$ (25)

Where, The operators $T_i: K_* \longrightarrow \mathcal{C}_+([0,1])$, are defined for each $i = 1,2$ by

$$T_1(u,v)(t) = \int_0^1 U_1(t,s)\, f(s,u(s),v(s))ds + \int_0^1 V_1(t,s)\, g(s,u(s),v(s))ds \quad (26)$$

$$T_2(u,v)(t) = \int_0^1 U_2(t,s)\, f\big(s,u(s),v(s)\big)ds + \int_0^1 V_2(t,s)\, g\big(s,u(s),v(s)\big)ds \quad (27)$$

Proposition 1. $T: K_* \longrightarrow K$given by$T(u,v) = \big(T_1(u,v), T_2(u,v)\big)$is well definedon$K_*$.

Proof. Let$(u,v) \in K_*$,of the estimate (20) for each$i = 1,2$to the following operator

$$T_i(u,v)(t) = \int_0^1 U_i(t,s)\, f(s,u(s),v(s))ds + \int_0^1 V_i(t,s)\, g(s,u(s),v(s))ds \ \le$$

$$\le \rho \int_0^1 f(s,u(s),v(s))\, ds + \rho \int_0^1 g(s,u(s),v(s))\, ds$$

Now, using the estimate (22), in the previous inequality, we obtain

$$T_i(u,v)(t) \le \rho \int_0^1 a^{\lambda_1} b^{\mu_1} f(s,1,1-s)\, ds + \rho \int_0^1 a^{\lambda_2} b^{\mu_2} g(s,1-s,1)\, ds < +\infty$$

Let$B_r(0) = \{(u,v) \in K\, ; \|(u,v)\|_1 < r\}$the open ball in$K$, of center in zero and radius$r > 0$.

Proposition 2. Let$0 < r < R$. The operator $T: \overline{B_R(0)} \backslash B_r(0) \longrightarrow K$ verifies$T\left(\overline{B_R(0)} \backslash B_r(0)\right) \subset K$.

Proof. Let$(u,v) \in \overline{B_R(0)} \backslash B_r(0)$, for every$t \in [0,1]$, by using(23)

$$T_1(u,v)(t) = \int_0^1 U_1(t,s)\, f(s,u(s),v(s))ds + \int_0^1 V_1(t,s)\, g(s,u(s),v(s))ds \ge$$

$$\ge \int_0^1 \eta(1-t) U_1(\tau,s)\, f\left(s,u(s),v(s)\right)ds + \int_0^1 \eta(1-t) V_1(\tau,s)\, g\big(s,u(s),v(s)\big)ds =$$

$=\eta(1-t)T_1(u,v)(\tau)$, para todo $\tau \in [0,1] \Longrightarrow T_1(u,v)(t) \ge \eta(1-t)\|T_1(u,v)\|$. From (24)

$$T_1(u,v)(t) = \int_0^1 U_1(t,s)\, f(s,u(s),v(s))ds + \int_0^1 V_1(t,s)\, g(s,u(s),v(s))ds \ge$$

$$\ge \int_0^1 \eta(1-t) V_2(\tau,s)\, f\left(s,u(s),v(s)\right)ds + \int_0^1 \eta(1-t) U_2(\tau,s)\, g\big(s,u(s),v(s)\big)ds =$$

$=\eta(1-t)T_2(u,v)(\tau)$, para todo $\tau \in [0,1] \Longrightarrow T_1(u,v)(t) \geq \eta(1-t)\|T_2(u,v)\| \Longrightarrow$

$T_1(u,v)(t) \geq \eta(1-t)\|(T_1(u,v),T_2(u,v))\|_1, T_2(u,v)(t) \geq \eta(1-t)\|(T_1(u,v),T_2(u,v))\|_1$

Then, by definition of $K \Longrightarrow T(u,v) = ((T_1(u,v),T_2(u,v)) \in K$, that is, $T\left(\overline{B_R(0)}\backslash B_r(0)\right) \subset K$.

Proposition 3. Let $0 < r < R$. $T: \overline{B_R(0)}\backslash B_r(0) \longrightarrow K$ is a continuous operator.

Proof. It is enough to show that $T_1, T_2: \overline{B_R(0)}\backslash B_r(0) \longrightarrow C_+([0,1])$ are continuous operators.

Let $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \overline{B_R(0)}\backslash B_r(0)$ be a sequence, such that $(u_n,v_n) \longrightarrow (u,v) \in \overline{B_R(0)}\backslash B_r(0) \implies$

$$\|(u_n,v_n)-(u,v)\|_1 \to 0 \quad , \qquad n\to\infty$$

$$T_1(u_n,v_n)(t) - T_1(u,v)(t) = \int_0^1 U_1(t,s)\, f\big(s,u_n(s),v_n(s)\big)ds + \int_0^1 V_1(t,s)\, g\big(s,u_n(s),v_n(s)\big)ds -$$

$$-\int_0^1 U_1(t,s)\, f\big(s,u(s),v(s)\big)ds - \int_0^1 V_1(t,s)\, g\big(s,u(s),v(s)\big)ds =$$

$$= \int_0^1 U_1(t,s)\left[f\big(s,u_n(s),v_n(s)\big) - f\big(s,u(s),v(s)\big)\right]ds$$

$$+\int_0^1 V_1(t,s)\left[g\big(s,u_n(s),v_n(s)\big) - g\big(s,u(s),v(s)\big)\right]ds$$

Hence, of the estimate **E1**, it yields $|T_1(u_n,v_n)(t) - T_1(u,v)(t)| \leq$

$$\leq \int_0^1 \rho\left|f\big(s,u_n(s),v_n(s)\big) - f\big(s,u(s),v(s)\big)\right|ds + \int_0^1 \rho\left|g\big(s,u_n(s),v_n(s)\big) - g\big(s,u(s),v(s)\big)\right|ds$$

Let $M = \sup\{\|(u,v)\|_1, \|(u_n,v_n)\|\}_{n\in\mathbb{N}}$, then, there exists a positive constant $a > 0$ such that

$M/a < 1$, letting $b = a/\eta r$. By using the estimate **E3** with constants a and b

$$\left|f\big(s,u_n(s),v_n(s)\big) - f\big(s,u(s),v(s)\big)\right| \leq 2a^{\lambda_1}b^{\mu_1}f(s,1,1-s) = \frac{2a^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}}f(s,1,1-s)$$

$$\left|g\big(s,u_n(s),v_n(s)\big) - g\big(s,u(s),v(s)\big)\right| \leq \frac{2a^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}}g(s,1-s,1)$$

Since (u_n,v_n) , $(u,v) \in \overline{B_R(0)}\backslash B_r(0)$. We can choose $0 < \delta < 1/2$, $r > 0$, such that

$u_n(t) \geq \eta r\delta$, $v_n(t) \geq \eta r\delta$, $u(t) \geq \eta r\delta$, $v(t) \geq \eta r\delta$, for all $t \in [\delta, 1-\delta]$. Because

$f\big(s,u(s),v(s)\big)$, $g\big(s,u(s),v(s)\big)$ is uniformly continuous on $[\delta,1-\delta] \times [\eta r\delta, R] \times [\eta r\delta, R]$, so

$$\lim_{n\to\infty}\left|f\big(s,u_n(s),v_n(s)\big)-f\big(s,u(s),v(s)\big)\right|=0,\ \lim_{n\to\infty}\left|g\big(s,u_n(s),v_n(s)\big)-g\big(s,u(s),v(s)\big)\right|=0$$

Uniformly on $s\in[\delta,1-\delta]$, From the Lebesgue dominated convergence theorem it follows that

$$\int_{\delta}^{1-\delta}\left|f\big(s,u_n(s),v_n(s)\big)-f\big(s,u(s),v(s)\big)\right|ds\longrightarrow 0,\int_{\delta}^{1-\delta}\left|g\big(s,u_n(s),v_n(s)\big)-g\big(s,u(s),v(s)\big)\right|ds\longrightarrow 0$$

For every$\varepsilon>0$ y $0<\delta<1/2$from the hypotheses (H1) (7) and(10).

$$\int_{[0,\delta]\cup[1-\delta,1]}\left|f\big(s,u_n(s),v_n(s)\big)-f\big(s,u(s),v(s)\big)\right|ds\le\int_{[0,\delta]\cup[1-\delta,1]}\frac{a^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}}f(s,1,1-s)\,ds<\varepsilon/4$$

$$\int_{[0,\delta]\cup[1-\delta,1]}\left|g\big(s,u_n(s),v_n(s)\big)-g\big(s,u(s),v(s)\big)\right|ds\le\int_{[0,\delta]\cup[1-\delta,1]}\frac{a^{\lambda_2+\mu_2}}{(\eta r)^{\mu_2}}g(s,1-s,1)\,ds<\varepsilon/4$$

For every$\varepsilon>0$, $\forall\exists\ n_0\in\mathbb{N}$such that$n\ge n_0$we have

$$\int_{\delta}^{1-\delta}\left|f\big(s,u_n(s),v_n(s)\big)-f\big(s,u(s),v(s)\big)\right|ds<\varepsilon/4,\int_{\delta}^{1-\delta}\left|g\big(s,u_n(s),v_n(s)\big)-g\big(s,u(s),v(s)\big)\right|ds<\varepsilon/4$$

$\Longrightarrow\|T_1(u_n,v_n)-T_1(u,v)\|\le\varepsilon.T_1(u,v)$ is a continuous operator. With the same arguments it shows that $T_2(u,v)$ is a continuous operatoron$\overline{B_R(0)}\backslash B_r(0)$.

Proposition 4. Let$0<r<R$. $T:\overline{B_R(0)}\backslash B_r(0)\longrightarrow K$is a compact operator.

Proof. Let$B\subset\overline{B_R(0)}\backslash B_r(0)$be a bounded subset, it must be shown that$T(B)$ is relatively compact.

Assertion 1 :$T(B)$ is uniformly bounded.

Since$B\subset\overline{B_R(0)}\backslash B_r(0)$is a bounded subset, $\Longrightarrow\exists\ M>0$, such that$\|(u,v)\|_1\le M,\forall(u,v)\in B$

Then, by the Proposition 1, for$a=M$,$b=M/\eta r$ and each$i=1,2$,we obtain

$$T_i(u,v)(t)\le\rho\int_0^1\frac{M^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}}f(s,1,1-s)\,ds+\rho\int_0^1\frac{M^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}}g(s,1-s,1)\,ds=$$

$$=\rho\frac{M^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}}\left[\int_0^1 f(s,1,1-s)\,ds+\int_0^1 g(s,1-s,1)\,ds\right]$$

Once again from (H1)(7) and (10) applied to the two integrals , we deduce that

$$T_i(u,v)(t)\le\rho\frac{M^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}}\tilde{C}\le C\quad\Longrightarrow\quad\|T_i(u,v)\|\le C\ ,i=1,2\ ,\forall\ (u,v)\in B\ \Longrightarrow$$

$\|T(u,v)\|_1=\max\limits_{1\le i\le 2}\|T_i(u,v)\|\le C\ \Longrightarrow T(B)$ is uniformly bounded.

<u>Assertion 2</u> **:**$T(B)$ is an equicontinuous operator. We prove$T_i(u,v)$ has a bounded derivative.

$$T_1(u,v)(t)=\int_0^1 U_1(t,s)\,f\big(s,u(s),v(s)\big)ds+\int_0^1 V_1(t,s)\,g\big(s,u(s),v(s)\big)ds$$

By substituting$U_1(t,s)$ and$V_1(t,s)$,we obtain

$$T_1(u,v)(t)=\frac{(1-t)\xi_1}{\xi}\int_0^1\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t)\,f\big(s,u(s),v(s)\big)ds+$$

$$+\int_0^1 G(t,s)\,f\big(s,u(s),v(s)\big)ds+\frac{(1-t)}{\xi}\int_0^1\int_0^1 \sigma_1(t)G(t,s)\,d\alpha(t)\,g\big(s,u(s),v(s)\big)ds$$

By the definition of Green's function$G(t,s)$

$$\int_0^1 G(t,s)\,f\big(s,u(s),v(s)\big)ds=\int_0^t s(1-t)\,f\big(s,u(s),v(s)\big)ds+\int_t^1 t(1-s)\,f\big(s,u(s),v(s)\big)ds$$

Deriving with respect to the variable $t$and by Leibnitz's rule, we have

$$\left(\int_0^1 G(t,s)\,f\big(s,u(s),v(s)\big)ds\right)'=-\int_0^t s\,f\big(s,u(s),v(s)\big)ds+\int_t^1(1-s)\,f\big(s,u(s),v(s)\big)ds$$

Then, deriving of the same way the operator $T_1(u,v)(t)$ with respect to t.

$$T_1'(u,v)(t)=\frac{-\xi_1}{\xi}\int_0^1\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t)\,f\big(s,u(s),v(s)\big)ds-\int_0^t s\,f\big(s,u(s),v(s)\big)ds$$

$$\int_t^1(1-s)\,f\big(s,u(s),v(s)\big)ds-\frac{1}{\xi}\int_0^1\int_0^1 \sigma_1(t)G(t,s)\,d\alpha(t)\,g\big(s,u(s),v(s)\big)ds$$

Taking the absolute value on both sides of the equality, and applying Fubini

$$|T_1'(u,v)(t)|\le\frac{\xi_1}{\xi}\int_0^1 f\big(s,u(s),v(s)\big)ds\int_0 \sigma_2(t)G(t,s)\,d\beta(t)+\int_0^t s\,f\big(s,u(s),v(s)\big)ds+$$

$$+\int_t^1(1-s)\,f\big(s,u(s),v(s)\big)ds+\frac{1}{\xi}\int_0^1 g\big(s,u(s),v(s)\big)ds\int_0^1 \sigma_1(t)G(t,s)\,d\alpha(t)$$

Sinceσ_1 y $\sigma_2$are functioins of $L^\infty(0,1)$, there are positive constantsc_1 y $c_2$such that

$$\int_0^1 \sigma_2(t)G(t,s)\,d\beta(t)\le c_2\ ,\qquad \int_0^1 \sigma_1(t)G(t,s)\,d\alpha(t)\le c_1$$

from (H1) (7) and (10) applied to the previous inequality , we obtain

$$|T_1'(u,v)(t)| \leq \frac{c_2\xi_1}{\xi}\rho\int_0^1 \frac{a^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}} f(s,1,1-s)\,ds + \int_0^t s\frac{a^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}} f(s,1,1-s)ds +$$

$$\int_t^1 (1-s)\frac{a^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}} f(s,1,1-s)ds + \frac{c_1}{\xi}\rho\int_0^1 \frac{a^{\lambda_2+\mu_2}}{(\eta r)^{\mu_2}} g(s,1-s,1)\,ds$$

$$|T_1'(u,v)(t)| \leq \frac{a^{\lambda_1+\mu_1}}{(\eta r)^{\mu_1}}\left(\frac{\rho c_2\xi_1}{\xi}\int_0^1 f(s,1,1-s)\,ds + \int_0^t s\,f(s,1,1-s)ds + \int_t^1 (1-s)\,f(s,1,1-s)ds\right) +$$

$+\frac{\rho a^{\lambda_2+\mu_2}}{(\eta r)^{\mu_2}}\frac{c_1}{\xi}\int_0^1 g(s,1-s,1)\,ds = \left|T_1'(u,v)(t)\right| \leq C \implies T_1$ is equicontinuous

In the same way, we get $|T_2'(u,v)(t)| \leq C \implies T_2$ is equicontinuous. Therefore by Arzelá-Ascoli's theorem, we deduce that $T(B)$ is relatively compact.

Proposition 5. Let (u,v) a positive solution of the system of second order differential equations (4)-(6). Then, there exists $a \in \mathbb{R}$, such that $0 < a < 1$ and it holds

$$\begin{cases} a(1-t) \leq u(t) \leq \frac{1}{a}(1-t) \\ a(1-t) \leq v(t) \leq \frac{1}{a}(1-t) \end{cases} \quad , t \in [0,1]$$

Proof. By the proposition 4, one has $(u,v) \in K \setminus \{0\}$, consequently

$$\eta\|(u,v)\|_1(1-t) \leq u(t) \quad , \quad v(t) \leq \|(u,v)\|_1$$

By choosing $a = \min\left\{\eta\|(u,v)\|_1, \frac{1}{c}, 1/2\right\} \implies a(1-t) \leq \eta\|(u,v)\|_1(1-t) \leq u(t)$

Let b be a constant such that $\|(u,v)\|_1/b < 1$ and $b > \frac{1}{\eta}$. From the proposition 3, we have

$$u(t) \leq \rho(1-t)\int_0^1 f\left(s,b,\frac{\eta\|(u,v)\|_1}{b}(1-s)\right)ds + \rho(1-t)\int_0^1 g\left(s,\frac{\eta\|(u,v)\|_1}{b}(1-s),b\right)ds \leq$$

$$\leq \frac{b^{\lambda_1+\mu_1}}{(\eta\|(u,v)\|_1)^{\mu_1}}\rho(1-t)\int_0^1 f(s,1,1-s)\,ds + \frac{b^{\lambda_2+\mu_2}}{(\eta\|(u,v)\|_1)^{\mu_2}}\rho(1-t)\int_0^1 g(s,1-s,1)\,ds = c(1-t)$$

From the previous inequality, we get $u(t) \leq \frac{1}{a}(1-t) \quad , t \in [0,1]$

With the same arguments, we show that $a(1-t) \leq v(t) \leq \frac{1}{a}(1-t) \quad , \quad t \in [0,1]$

Theorem 1 (Compression and expansion fixed point theorem)

Let K be a cone , Ω_1 , Ω_2 two open and bounded subsets of a Banach space

E, such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and $T: K \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow K$ a compact operator. Suppose one of the conditions is true

C1) $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_2$ or

C2) $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_2$

Then T has at least a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.See Dajun,G. and Lakshmikanthan,V.(1988).

Theorem 2. Suppose that $f \in C^0\big((0,1) \times [0,+\infty) \times (0,+\infty), [0,+\infty)\big)$,

$g \in C^0\big((0,1) \times (0,+\infty) \times [0,+\infty), [0,+\infty\big)$ satisfy the hypotheses (H1) and let

$\xi_1 = \int_0^1 (1-t)\sigma_1(t)\, d\alpha(t)$, $\xi_2 = \int_0^1 (1-t)\sigma_2(t)\, d\beta(t)$ for σ_1 ,σ_2 in $L^\infty(0,1)$ such that $\xi_1 > 0$, $\xi_2 > 0$, furthermore $\xi = 1 - \xi_1.\xi_2 > 0$ and $\lambda_1 + \lambda_2 < 1$, $\mu_1 + \mu_2 < 1$. Then there is only one positive solution (u, v) of the differential system (4)-(6).

Proof. i)Existence. Let $0 < r < 1 < R$. By the proposition 4, the continuous operator $T: \overline{B_R(0)} \backslash B_r(0) \longrightarrow K$ isa compact operator. By extension theorem of Dugundji(1951), we deduce that the operator $T: \overline{B_R(0)} \longrightarrow K$ is continuous and so that, it turns out a compact operator. Let $(u, v) \in \partial P_r \implies r = \|(u, v)\|_1$,using the inequality(19) in

$U_1(t,s).U_1(t,s) = \frac{(1-t)\xi_1}{\xi} \int_0^1 \sigma_2(t) G(t,s)\, d\beta(t) + G(t,s) \geq$

$$\geq \frac{(1-t)\xi_1}{\xi} \int_0^1 \sigma_2(t) t(1-t)s(1-s)\, d\beta(t) + t(1-t)s(1-s) \geq \nu(1-t)s(1-s)$$

$$U_2(t,s) = \frac{(1-t)}{\xi} \int_0^1 \sigma_1(t) G(t,s)\, d\alpha(t) \geq \frac{(1-t)}{\xi} \int_0^1 \sigma_1(t) t(1-t)s(1-s)\, d\alpha(t) \geq$$

$$\geq \nu(1-t)s(1-s)$$

Hence, for all $t \in [0,3/4]$, and each $i = 1,2$, we obtain:

$$T_i(u,v)(t) \geq \int_0^1 \nu(1-t)s(1-s)\, f(s,u(s),v(s))ds \;\geq \frac{\nu}{4}\int_0^1 s(1-s)\, f(s,u(s),v(s))ds$$

Since$(u,v) \in \partial P_r$we have $\eta r(1-t) \leq u(t)\;, v(t) \leq r$. Then, for every$t \in [0,1]$ and$r \leq \frac{1}{2} < 1$.

From the hypotheses about f, the previous inequality it turns out

$$T_i(u,v)(t) \geq \frac{\nu}{4}\int_0^1 s(1-s)\, f(s,\eta r(1-s),1)ds$$

By using the consequence 1 to the function$f$

$$T_i(u,v)(t) \geq \frac{\nu}{4}(\eta r)^{\lambda_1}\int_0^1 s(1-s)\, f(s,1-s,1)ds =$$

$= \left(\frac{\nu}{4}\eta^{\lambda_1}\int_0^1 s(1-s)\, f(s,1-s,1)ds\right) r^{-(1-\lambda_1)}.r \geq r = \|(u,v)\|_1$, for all r such that

$$0 < r \leq \min\left\{\left(\frac{\nu}{4}\eta^{\lambda_1}\int_0^1 s(1-s)\, f(s,1-s,1)ds\right)^{\frac{1}{1-\lambda_1}}, \frac{1}{2}\right\}$$

Let$(u,v) \in \partial P_R \quad \Longrightarrow R = \|(u,v)\|_1$,using (20) for each $i = 1,2$to the following operator

$$T_i(u,v)(t) = \int_0^1 U_i(t,s)\, f(s,u(s),v(s))ds + \int_0^1 V_i(t,s)\, g(s,u(s),v(s))ds \;\leq$$

$$\leq \rho\int_0^1 f(s,u(s),v(s))\, ds + \rho\int_0^1 g(s,u(s),v(s))\, ds$$

Because$(u,v) \in \partial P_R$we have$\eta R(1-t) \leq u(t)\,,\; v(t) \leq R$, for all$t \in [0,1]$

By using the hypotheses (H1) (8) and the consequence 2 (11) to f and$g$ respectively

$$T_i(u,v)(t) \leq \rho\int_0^1 f(s,R,\eta R(1-s))\, ds + \rho\int_0^1 g(s,\eta R(1-s),R)\, ds \leq$$

$$\leq \rho(\eta R)^{-\mu_1}\int_0^1 f(s,R,(1-s))\, ds + \rho(\eta R)^{-\mu_2}\int_0^1 g(s,(1-s),R)\, ds$$

For$R \geq 1/\eta$, it yields

$$T_i(u,v)(t) \leq \rho\int_0^1 f(s,R,(1-s))\, ds + \rho\int_0^1 g(s,(1-s),R)\, ds$$

For$R \geq 2 > 1$, applying the consequence 1 and 3to$f$and , we have

$$T_i(u,v)(t) \leq \rho R^{\lambda_1} \int_0^1 f(s,1,(1-s))\,ds + \rho R^{\lambda_2} \int_0^1 g(s,(1-s),1)\,ds \leq$$

$$\leq 1.\rho R^{\max\{\lambda_1,\lambda_2\}} \left(\int_0^1 f\big(s,1,(1-s)\big)\,ds + \int_0^1 g(s,(1-s),1)\,ds \right) =$$

$\leq R.\rho R^{-(1-\max\{\lambda_1,\lambda_2\})} \left(\int_0^1 f\big(s,1,(1-s)\big)\,ds + \int_0^1 g(s,(1-s),1)\,ds\right) \leq R = \|(u,v)\|_1$. For R

$$R \geq \max\left\{ \left(\rho \int_0^1 f\big(s,1,(1-s)\big)\,ds + \rho ds \right)^{\frac{1}{1-max\{\lambda_1,\lambda_2\}}}, \frac{1}{\eta}, 2 \right\}$$

Hence, applying the Theorem 1, the operator $T: \overline{B_R(0)} \backslash B_r(0) \longrightarrow K$has a fixed point$(u,v)$in

$\overline{B_R(0)} \backslash B_r(0)$and so that one gets a positive solution in $\overline{B_R(0)} \backslash B_r(0)$.

ii) Uniqueness. Suppose that(u_1,v_1) , $(u_2,v_2)$are two positive solutions in $\overline{B_R(0)} \backslash B_r(0)$of the system (4)-(6). By the proposition 5**.** There exists$a,b \in \mathbb{R}$, $(a,b) \in (0,1) \times (0,1)$it holds

$$\begin{cases} a(1-t) \leq u_1(t), \leq \frac{1}{a}(1-t) \\ a(1-t) \leq v_1(t) \leq \frac{1}{a}(1-t) \end{cases} \quad ,t \in [0,1]$$

$$\begin{cases} b(1-t) \leq u_2(t) \leq \frac{1}{b}(1-t) \\ b(1-t) \leq v_2(t) \leq \frac{1}{b}(1-t) \end{cases} \quad ,t \in [0,1]$$

Hence, $u_2(t) \leq \frac{1}{b}(1-t) = \frac{a}{ba}(1-t) \leq \frac{1}{ba} u(t)$

$u_2(t) \geq b(1-t) = \frac{ba}{a}(1-t) \geq bau_1(t) \Longrightarrow abu_1(t) \leq u_2(t) \leq \frac{1}{ab} u_1(t)$. In the same way

$abv_1(t) \leq v_2(t) \leq \frac{1}{ab} v_1(t)$ .Since , $a,b \neq 1$, letting

$$S = \sup\left\{a\,;\; au_1(t) \leq u_2(t) \leq \frac{1}{a}u_1(t)\,, av_1(t) \leq v_2(t) \leq \frac{1}{a}v_1(t)\,, t \in [0,1]\right\}$$

From the definition of supremum, we have $0 < ab \leq S < 1$, moreover

$Su_1(t) \leq u_2(t) \leq \frac{1}{S}u_1(t)$, $Sv_1(t) \leq v_2(t) \leq \frac{1}{S}v_1(t), t \in [0,1]$. Using H1, it yields

$$f(t, u_2(t), v_2(t)) \geq f(t, Su_1(t), \frac{1}{S}v_1(t)) \geq S^{\lambda_1+\mu_1} f(t, u_1(t), v_1(t)) \geq S^{\lambda} f(t, u_1(t), v_1(t))$$

Where, $= \max\{\lambda_1 + \mu_1, \lambda_2 + \mu_2\}$, such that$\lambda < 1$, we obtain

$$u_2(t) = T(u_2, v_2)(t) = \int_0^1 U_1(t,s)\, f(s, u_2(s), v_2(s))ds + \int_0^1 V_1(t,s)\, g(s, u_2(s), v_2(s))ds \geq$$

$$\geq S^{\lambda}\left(\int_0^1 U_1(t,s)\, f\big(s, u_1(s), v_1(s)\big)ds + \int_0^1 V_1(t,s)\, g\big(s, u_1(s), v_1(s)\big)ds\right) =$$

$= S^{\lambda}T(u_1, v_1)(t) = S^{\lambda}u_1(t)$.With the same arguments, we obtain

$v_2(t) \geq S^{\lambda}v_1(t)$, $u_1(t) \geq S^{\lambda}u_2(t), v_1(t) \geq S^{\lambda}v_2(t)$ .Note that$(S, \lambda) \in (0,1) \times (0,1)$ so that,

one obtains a contradiction with the supremum S.Therefore we get the result.

4. CONCLUSION

We have achieved a result of existence and uniqueness through the Compression and expansion fixed point theorem.

5. REFERENCES

Asif, N. A., y Khan, R. A. (2012). Positive solutions to singular system with four-point coupled boundary conditions. Journal of Mathematical Analysis and Applications, 386(2), 848-861.

Asif, N. A., Khan, R. A., y Henderson, J. (2010). Existence of positive solutions to a system of singular boundary value problems. Dynamic Systems and Applications, 19(2), 395-404.

Botelho, G., Pellegrino, D. y Teixeira, E. (2015). Fundamentos del análisis funcional. SBM. Cheng, X., y Zhong, C. (2005). Existence of positive solutions for a second-order ordinary differential system. Journal of Mathematical Analysis and Applications,312(1), 14-23.

Cui, Y., Liu, L., y Zhang, X. (2013). Uniqueness and existence of positive solutions for singular differential systems with coupled integral boundary value problems. In Abstract and Applied Analysis (Vol. 2013). Hindawi Publishing Corporation.

Cortez,M. , Quique, J.(2023)Existência de solução de um sistema acoplado elíptico não-linear

Brazilian Journal of Development, Curitiba, v.9, n.7, p. 22161-22171

Cassius Vinícius Casaro, Diego Márcio Gonzaga, Marco Rogério da Silva Richetto, Kátia Celina da Silva Richetto.(2022) Implementação de automatização do processo de fabricação de polímero utilizando a plataforma SDCD YokogawaCentum VP, BrazilianJournalofdevelopment Vol. 8 No. 12.

Cui, Y., y Sun, J. (2012). On existence of positive solutions of coupled integral boundary value problems for a nonlinear singular superlinear differential system.Electronic Journal of Qualitative Theory of Differential Equations, 2012(41), 1- 13.

Cui, Y., y Zou, Y. (2015).An existence and uniqueness theorem for a second order nonlinear system with coupled integral boundary value conditions. Applied Mathematics and computation, 2015(256), 438- 444.

Dajun,G. and Lakshmikanthan,V.(1988), Nonlinear problems in abstract cones, Academic Press, Inc.N.Y.

Doering, C. y Lopes A. (2016). Equações diferenciaisordinárias. Instituto de matemática pura e aplicada-IMPA. (6ª Ed.). IMPA, Rio de Janeiro, Brasil. 423 p.

Dugundji, J.(1951) An extension of Tietze's Theorem, Pacific J. Math., 1, 353-367.

Liu, B., Liu, L., y Wu, Y. (2008). Positive solutions for a singular second-order three point boundary value problem. Applied Mathematics and Computation, 196(2), 532- 541.

Lu¨, H., Yu, H., y Liu, Y. (2005). Positive solutions for singular boundary value problems of a coupled system of differential equations. Journal of Mathematical Analysis and Applications, 302(1), 14-29