e-ISSN 3050-9645
p-ISSN 2795-3274

GPH INTERNATIONAL JOURNAL OF MATHEMATICS
WWW.GPHJOURNAL.ORG
10.5281/zenodo.16355788 Vol. 8 Issue 05 May - 2025 Manuscript ID: #2017

Generalized solutions of first order nonlinear Cauchy problems

Dennis F. Agbebaku[1], Celestine B. Agbata[2], Chioma L. Ejikeme[3] and Mary B. Okofu[4]
[1]Department of Mathematics, University of Nigeria, 410001 Nsukka, Nigeria. dennis.agbebaku@unn.edu.ng
[2]Department of Mathematics and Statistics, Faculty of Science, Confluence University of Science and Technology, Osara, Nigeria agbatacelestine92@gmail.com
[3]Department of Mathematics, University of Nigeria, Nsukka. chioma.ejikeme@unn.edu.ng
[4]Department of Mathematics University of Nigeria, Nsukka. mary.okofu@unn.edu.ng

Abstract

In this paper, we show how the Order Completion Method for systems of nonlinear partial differential equations may be applied to solve first order initial value problems. In particular, we construct generalized solutions of a large family of such initial value problems in two related spaces of generalized functions. The way in which the two mentioned solution concepts relate to each other is discussed, as well as the basic regularity properties of solutions.

How to cite: Agbebaku, D., Agbata, C., Ejikeme, C., & Okofu, M. (2025). Generalized solutions of first order nonlinear Cauchy problems. GPH-International Journal of Mathematics, 8(5), 32-54. https://doi.org/10.5281/zenodo.16355788

1 Introduction

It is a widely held belief among mathematicians specializing in nonlinear partial differential equations (PDEs) that a general and type independent theory for the existence and basic regularity properties of generalized solutions of such equations is not possible. Indeed, the book [4] by I. V. Arnold starts on the first page with the following statement:

> "In contrast to ordinary differential equations, there is no unified theory of partial differential equations. Some equations have their own theories, while others have no theory at all. The reason for this complexity is a more complicated geometry ..."

However, the perceived inability of mathematics to deal with PDEs in a unified way should be attributed to the inherent limitations of the customary, linear topological theories for the solution of PDEs themselves, rather than to any fundamental conceptual obstacles.

In this regard, let us mention only the following. A nonlinear analytic PDE will, according to the well known Cauchy-Kowalevskaia Theorem [11], admit an analytic solution which is defined on a neighborhood of any non-characteristic hyper surface on which analytic initial data is specified. However, outside of this neighborhood the solution may fail to exist. In particular, the solution will typically exhibit singularities outside the mentioned neighborhood of analyticity. On the other hand, the spaces of generalized functions that are typically used in the study of solutions of linear and nonlinear PDEs cannot deal with large classes of singularities. Indeed, due to the celebrated Sobolev Embedding Theorem, none of the Sobolev spaces can deal with the most simple singular functions, such as the Heaviside step function.

$$H(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

Furthermore, Colombeau generalized functions [8], and therefore distributions as well, cannot handle an analytic function with an essential singularity at one single point, such as $f(z) = e^{1/z}$.

Recall that, due the the Great Picard Theorem in complex analysis, any analytic function with an essential singularity at a point $c \in \mathbb{C}$ will assume every complex number, with possibly one exception, infinitely many times as a value in every neighborhood of the point c. Such a singular function will clearly violate the polynomial type growth conditions which are, rather as a rule, imposed on generalized functions.

In contradistinction with the mentioned usual methods for the solutions of PDEs, two recent theories provide general and type independent results regarding the existence and basic regularity properties of large classes.

paces of (piecewise) smooth functions, and applies to what may be considered all continuous nonlinear PDEs. Furthermore, the solution so obtained satisfies a blanket regularity property. In particular, the solutions may be assimilated with Hausdorff continuous interval functions [3]. Based on the recent reformulation and enrichment of the OCM in terms of suitable uniform convergence spaces and their completions, the regularity properties of solutions, as well as the understanding of the structure of solutions, have been significantly improved, [19, 20, 21, 22].

Neuberger [12, 13, 14, 15] introduced a solution technique for nonlinear PDEs, which is based on a generalized method of Steepest Descent in suitably constructed Hilbert spaces. The underlying ideas uppon which the theory is based does not depend on the particular form of the PDE involved, and is therefore type independent. However, the relevant techniques involve several highly technical aspects which have, as of yet, not been resolved for a class of equations comparable to that to which the OCM applies. However, the numerical computation of solutions, based on this theory, has advanced beyond the proven scope of the underlying analytical techniques. In this regard, remarkable results have been obtained, see for instance [15].

In this paper we apply the OCM, as formulated in the context of uniform convergence spaces [20,21,22] to the first order nonlinear Cauchy problem

$$D_t u(x,t) + F(x,t,u(x,t),D_x u(x,t)) = f(x,t), \tag{1.1}$$

$$u(x,0) = u_0(x), x \in (-a,a). \tag{1.2}$$

Here $(x,t) \in \Omega = (-a,a) \times (-b,b) \subseteq \mathbb{R}^2$ for some $a, b > 0$, while

$$F: \bar{\Omega} \times \mathbb{R}^4 \longrightarrow \mathbb{R}$$

is jointly continuous in all its variables. The initial value u_0 is assumed to be $\mathcal{C}^1$-smooth on $[-a, a]$.

2 Function Spaces and their Completion

As is shown in [19,21] the PDE (1.1) admits generalized solutions, both in the pull-back type space of generalized functions $\mathcal{NL}_T^1(\Omega)$ introduced in [19], and in the Sobolev type space of generalized functions $\mathcal{NL}^1(\Omega)$, considered in [21]. However, such generalized solutions may fail to satisfy the initial condition (1.2), in any suitable generalized sense. In fact, a generalized function in $\mathcal{NL}^1(\Omega)$ may not have a well defined trace in $\mathcal{NL}(\mathbb{R} \times \{0\})$. One should note that this is not a situation peculiar to the OCM, but rather a typical feature of solution methods for PDEs involving singular objects as generalized solutions of such equations.

The aim of this section is to introduce suitable modifications of the spaces $\mathcal{NL}^1(\Omega)$ and $\mathcal{NL}_T^1(\Omega)$ that can accommodate the presence of the initial condition (1.2). The construction of these spaces, as well as the arguments leading to the existence of solutions presented in Section 3 , follow by the same methods that apply to the free problem, see [19, 21]. In this way, we come to appreciate yet another advantage of solving linear and nonlinear PDEs through the OCM. Namely, that initial and / or boundary value problems may be solved by exactly the same techniques that apply to the free equation (1.1).

2.1 Function Spaces

Let $\mathcal{ML}_0^1(\Omega)$ denote the set

$$\mathcal{ML}_0^1(\Omega) = \left\{ u \in \mathcal{ML}^1(\Omega) \left| \begin{array}{l} \forall\, x \in [-a, a]: \\ 1)\ u, \mathcal{D}_x u \text{ are continuous and finite at } (x, 0) \\ 2)\ \mathcal{D}_x u(x, 0) = \dfrac{d}{dx} u(x, 0) \end{array} \right. \right\}, \tag{2.1}$$

while $\mathcal{ML}_0^0(\Omega)$ is defined as

$$\mathcal{ML}_0^0(\Omega) = \left\{ u \in \mathcal{ML}^0(\Omega) \left| \begin{array}{l} \forall \quad x \in [-a, a]: \\ \quad u \text{ is continuous and finite at } (x, 0) \end{array} \right. \right\}. \tag{2.2}$$

Here $\mathcal{D}_x$ denotes the differential operator defined by

$$\mathcal{D}_x u(x, t) = \Big(I\big(S(D_x u) \big) \Big) (x, t) \tag{2.3}$$

In view of (2.1) and it is clear that the relation (2.3) defines a mapping

$$\mathcal{D}_x : \mathcal{ML}_0^1(\Omega) \to \mathcal{ML}_0^0(\Omega), \tag{2.4}$$

while (2.1) and (2.2) implies the inclusion

$$\mathcal{ML}_0^1(\Omega) \subset \mathcal{ML}_0^0(\Omega) \tag{2.5}$$

Similarly, the expression

$$\mathcal{D}_t u(x,t) = \left(I\left(S(D_t u)\right)\right)(x,t)$$

defines a mapping

$$\mathcal{D}_t : \mathcal{ML}_0^1(\Omega) \to \mathcal{ML}^0(\Omega), \tag{2.6}$$

The mappings (2.4) and (2.6) are extensions of the usual linear partial differential operators D_x and D_t acting on spaces of smooth functions. Using these extended differential operators, we associate a mapping

$$T : \mathcal{ML}_0^1(\Omega) \to \mathcal{ML}^0(\Omega) \tag{2.7}$$

with the PDE (1.1) through the expression

$$Tu(x,t) = I\left(S\left(\mathcal{D}_t u + F(\cdot,\cdot,u,\mathcal{D}_x u)\right)\right)(x,t)$$

We may note that any classical solution $u \in \mathcal{C}^1(\Omega)$ of the PDE (1.1) will also satisfy equation

$$Tu = f \tag{2.8}$$

Thus we may view (2.8) as a first extension of the PDE (1.1). In order to also incorporate the initial value (1.2) into this first extension of the classical setup (1.1-1.2), we introduce the mapping

$$\bar{T} : \mathcal{ML}_0^1(\Omega) \ni u \mapsto (Tu, u_{|t=0}) \in \mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a] \tag{2.9}$$

In view of (2.1) the mapping (2.9) is well defined. The equation

$$\bar{T}u = (f, u_0) \tag{2.10}$$

is an extension of the initial value problem (1.1-1.2) in the following sense: Any solution $u \in \mathcal{ML}_0^1(\Omega)$ of (2.10) satisfies the extension (2.8) of the PDE (1.1) as well as the initial value (1.2). In this way, we may view the single equation (2.10) as a generalization of the initial value problem (1.1-1.2). Two further generalizations of the initial value problem (1.1-1.2) are obtained by extending the mapping (2.9) to spaces of generalized functions, the construction of which we now turn to.

2.2 Pullback Spaces

The general method underlying the construction of the pullback space of generalized functions $\mathcal{NL}_T^1(\Omega)$ presented in [19] consists of defining the structure on the space of generalized functions in terms of the differential operator T. In order to deal with the initial condition (1.2), we consider the mapping (2.9).

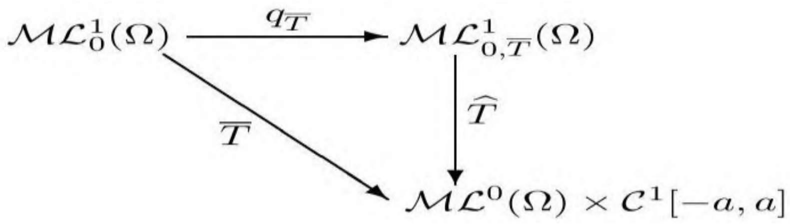
Define an equivalence relation $\sim_{\bar{T}}$ on $\mathcal{ML}_0^1(\Omega)$ through the relation

$$u \sim_{\bar{T}} v \Leftrightarrow \bar{T}u = \bar{T}v \tag{2.11}$$

and denote the quotient space $\mathcal{ML}_0^1(\Omega)/\sim_{\bar{T}}$ by $\mathcal{ML}_{0,\bar{T}}^1(\Omega)$. With the mapping (2.9) we may associate in a canonical way an injective mapping

$$\hat{T} : \mathcal{ML}_{0,\bar{T}}^1(\Omega) \to \mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a] \tag{2.12}$$

such that the diagram



commutes, with $q_{\bar{T}}$ denoting the canonical quotient mapping associated with the equivalence relation (2.11). Note that, in view of (2.13), a solution $U \in \mathcal{ML}^1_{0,\bar{T}}(\Omega)$ of the equation

$$\hat{T}U = (f, u_0) \tag{2.14}$$

is in fact the equivalence class, with respect to the equivalence relation (2.11), of all solutions of (2.10). Thus equation (2.14) is an equivalent formulation of (2.10). Thus we consider any extension of (2.14) as a generalization of equation (2.10), and hence of the initial value problem (1.1-1.2).

Consider on $\mathcal{ML}^0(\Omega)$ the uniform order convergence structure, see Definition ??, Appendix A, while $\mathcal{C}^1[-a,a]$ is equipped with the following uniform convergence structure.
Definition 2.1. A filter $\mathcal{U}$ on $\mathcal{C}^1[-a,a]$ belongs to $\mathcal{J}_{\mathrm{u}}$ if, for some $k \in \mathbb{N}$, we have

$$\exists \quad u_1, \dots, u_k \in \mathcal{C}^1[-a,a]:$$
$$\exists \quad \mathcal{F}_1, \dots, \mathcal{F}_k \text{ filters on } \mathcal{C}^1[-a,a]:$$

1. $\mathcal{F}_i$ converges uniformly to $u_i, i = 1, \dots, k$
2. $(\mathcal{F}_1 \times \mathcal{F}_1) \cap \dots \cap (\mathcal{F}_k \times \mathcal{F}_k) \subseteq \mathcal{U}$

The family of filters $\mathcal{J}_{\mathrm{u}}$ on $\mathcal{C}^1[-a,a] \times \mathcal{C}^1[-a,a]$ is trivially seen to be a Hausdorff and complete uniform convergence structure. Indeed, $\mathcal{J}_{\mathrm{u}}$ is the associated uniform convergence structure, see [7, Proposition 2.1.7], associated with the (topological) convergence structure of uniform convergence in $\mathcal{C}^1[-a,a]$.

The Cartesian product $\mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a]$ carries the product uniform convergence structure $\mathcal{J}_{o,\mathrm{u}}$, with the components equipped with the uniform order convergence structure $\mathcal{J}_o$ and the uniform convergence structure $\mathcal{J}_{\mathrm{u}}$, respectively. That is,

$$\mathcal{U} \in \mathcal{J}_{o,\mathrm{u}} \Leftrightarrow \begin{pmatrix} 1) & (\pi_1 \times \pi_1)(\mathcal{U}) \in \mathcal{J}_o \\ 2) & (\pi_2 \times \pi_2)(\mathcal{U}) \in \mathcal{J}_{\mathrm{u}} \end{pmatrix}. \tag{2.15}$$

Here π_1 and π_2 denote the projections

$$\pi_1 : \mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a] \ni (f,u) \mapsto f \in \mathcal{ML}^0(\Omega)$$

and

$$\pi_2 : \mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a] \ni (f,u) \mapsto u \in \mathcal{C}^1[-a,a]$$

As mentioned in Appendix A, the completion of $\mathcal{ML}^0(\Omega)$ with respect to the uniform convergence structure $\mathcal{J}_o$ is the space $\mathcal{NL}(\Omega)$ of all nearly finite normal lower semi-continuous functions on Ω, equipped with a suitable uniform convergence structure, defined in Definition ??. Thus, since $\mathcal{C}^1[-a,a]$ is complete with respect to $\mathcal{J}_{\mathrm{u}}$, it follows from [22, Theorem 3.2] that there exists a bijective uniformly continuous mapping

$$i^\sharp : (\mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a])^\sharp \to \mathcal{NL}(\Omega) \times \mathcal{C}^1[-a,a] \tag{2.16}$$

Here $\mathcal{NL}(\Omega) \times \mathcal{C}^1[-a,a]$ is equipped with the product uniform convergence structure, with $\mathcal{NL}(\Omega)$ and $\mathcal{C}^1[-a,a]$ equipped with the uniform convergence structures $\mathcal{J}_o^\sharp$ and $\mathcal{J}_{\mathrm{u}}$, respectively. In view of (2.16), we identify the completion $(\mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a])^\#$ of $\mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a]$ with the set $\mathcal{NL}(\Omega) \times \mathcal{C}^1[-a,a]$, equipped with a suitable uniform convergence structure. However, we should note that the mapping (2.16) does not have a continuous inverse. Thus the uniform convergence structure on $\mathcal{NL}(\Omega) \times \mathcal{C}^1[-a,a]$, when viewed as the completion of $\mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a]$ does not carry the product uniform convergence structure. Except if otherwise mentioned, we consider $\mathcal{NL}(\Omega) \times \mathcal{C}^1[-a,a]$ as the completion of $\mathcal{ML}^0(\Omega) \times \mathcal{C}^1[-a,a]$, thus equipped with a uniform convergence structure other than the product uniform convergence structure.

We equip $\mathcal{ML}^1_{0,\bar{T}}(\Omega)$ with the initial uniform convergence structure $\mathcal{J}_{\hat{T}}$ with respect to the mapping (2.12). That is,

$$\mathcal{U} \in \mathcal{J}_{\hat{T}} \Leftrightarrow \hat{T}(\mathcal{U}) \in \mathcal{J}_{o,\mathrm{u}} \tag{2.17}$$

Thus the mapping (2.12) is a uniformly continuous embedding. Therefore, in view of [22, Corollary 2.4], the unique uniformly continuous extension of $\hat{T}$,

$$\hat{T}^\sharp : \mathcal{NL}^1_{0,\bar{T}}(\Omega) \to \mathcal{NL}(\Omega) \times \mathcal{C}^1[-a,a] \tag{2.18}$$

is injective, where $\mathcal{NL}^1_{0,\bar{T}}(\Omega)$ denotes the completion of $\mathcal{ML}^1_{0,\bar{T}}(\Omega)$. In this way, we may identify $\mathcal{NL}^1_{0,\bar{T}}(\Omega)$ with a subset of $\mathcal{NL}(\Omega) \times \mathcal{C}^1[-a,a]$, thus providing a first blanket regularity for solutions of the generalized equation

$$\hat{T}^\sharp U^\sharp = (f, u_0). \tag{2.19}$$

Namely, any solution $U^\sharp \in \mathcal{NL}^1_{0,\bar{T}}(\Omega)$ of (2.19) may be assimilated with usual real nearly finite normal lower semi-continuous functions. Furthermore, the restriction of such a solution $U^\sharp$ of (2.19), given by the second component of the mapping (2.18), is equal to u_0, so that $U^\sharp$ satisfies also the initial condition (1.2) in this generalized sense.

2.3 Sobolev-type Spaces

The main advantage of considering generalized solutions of (1.1-1.2) in the context of the pullback space $\mathcal{NL}_{0,\bar{T}}(\Omega)$ introduced in Section 2.2, is the particularly simple and transparent way in which one arrives as the mentioned space of generalized functions. On the other hand, the drawback of this method is that the structure and properties of generalized solutions, beyond the basic regularity properties inherent in the construction, may be lost in the construction. It may be due to several issues, of which we mention the following. The space of generalized functions
is, to a good extent determined by the particular nonlinear partial differential operator T. Furthermore,

one encounters significant difficulties when trying to introduce additional structure, such as generalized derivatives, on spaces of generalized functions. These issues may be overcome by considering so called Sobolev-type spaces of generalized functions, the construction of which we now turn to.

In this regard, we introduce a suitable uniform convergence structure on $\mathcal{ML}_0^0(\Omega)$. We should recall that the constructions leading to the existence of generalized solutions of systems of nonlinear PDEs within the context of the Order Completion Method are based on concepts of approximation in terms of the pointwise order relation on spaces of extended real valued functions. In particular, the notion of order convergence plays a fundamental role in the methods introduced in [19, 20, 21, 22]. It is precisely these constructions are we now adapt to the case of the initial value problem (1.1-1.2). In view of these remarks, we introduce the following uniform convergence structure on $\mathcal{ML}_0^0(\Omega)$.

Definition 2.2. A filter $\mathcal{U}$ on $\mathcal{ML}_0^0(\Omega)$ belongs to $\mathcal{J}_1$ if, for some $k \in \mathbb{N}$, the following condition is satisfied:

$\forall j = 1, \ldots, k$:
$\exists (\lambda_n^j), (\mu_n^j) \subset \mathcal{ML}_0^0(\Omega)$:
$\exists u_j \in \mathcal{NL}(\Omega)$:

1) $\lambda_n^j \leq \lambda_{n+1}^j \leq \mu_{n+1}^j \leq \mu_n^j, n \in \mathbb{N}$ (2.20)
2) $\lambda_n^j(x,0) = \mu_n^j(x,0), x \in [-a,a], n \in \mathbb{N}$ (2.20)
3) $\sup\{\lambda_n^j : n \in \mathbb{N}\} = u_j = \inf\{\mu_n^j : n \in \mathbb{N}\}$ (2.20)
4) $[\{I_n^1 \times I_n^1 : n \in \mathbb{N}\}] \cap \ldots \cap [\{I_n^k \times I_n^k : n \in \mathbb{N}\}] \subseteq \mathcal{U}$(2.20)

Here I_n^j denotes the order interval $[\lambda_n^j, \mu_n^j]$.

Proposition 2.3. The family $\mathcal{J}_1$ is a Hausdorff uniform convergence structure on $\mathcal{ML}_0^0(\Omega)$.

Proof. We verify that $\mathcal{J}_1$ satisfies the axioms of a uniform convergence structure, see [7, Definition 2.1.2].

(i) Consider some $u \in \mathcal{ML}_0^0(\Omega)$. In Definition 2.2, set $k = 1$ and $\lambda_n^1 = \mu_n^1 = u$ for all $n \in \mathbb{N}$. Clearly conditions 1) to 3) of (2.20) are satisfied, and

$$[u] \times [u] = [\{I_n^1 \times I_n^1 : n \in \mathbb{N}\}]$$

Therefore $[u] \times [u] \in \mathcal{J}_1$.

(ii) Let $\mathcal{U}$ and $\mathcal{V}$ belong to $\mathcal{J}_1$, and let $(I_n^1), \ldots, (I_n^k)$ and $(J_n^1), \ldots, (J_n^l)$ be the sequences of order intervals associated with $\mathcal{U}$, respectively $\mathcal{V}$, through Definition 2.2. For $j = k+1, \ldots, k+l$, set

$$I_n^j = J_n^{j-k}, n \in \mathbb{N}.$$

Then it follows from (2.20) that

$$[\{I_n^1 \times I_n^1 : n \in \mathbb{N}\}] \cap \ldots \cap [\{I_n^{k+l} \times I_n^{k+l} : n \in \mathbb{N}\}] \subseteq \mathcal{U} \cap \mathcal{V}$$

Therefore $\mathcal{U} \cap \mathcal{V}$ belongs to $\mathcal{J}_1$.

(iii) It is trivially true that if $\mathcal{U} \in \mathcal{J}_1$, and $\mathcal{V} \supseteq \mathcal{U}$, then $\mathcal{V} \in \mathcal{J}_1$.

(iv) According to Definition 2.2, every filter $\mathcal{U} \in \mathcal{J}_1$ contains a filter $\mathcal{V} \in \mathcal{J}_1$ which has a symmetric basis. That is, $\mathcal{V}^{-1} = \mathcal{V} \in \mathcal{J}_1$. But, since $\mathcal{V} \subseteq \mathcal{U}$, it follows that $\mathcal{V}^{-1} \subseteq \mathcal{U}^{-1}$. Therefore $\mathcal{U}^{-1} \in \mathcal{J}_1$.

(v) Consider $\mathcal{U}, \mathcal{V} \in \mathcal{J}_1$ with the property that $\mathcal{U} \circ \mathcal{V}$ exists. Let $(I_n^1), \dots, (I_n^k)$ and $(J_n^1), \dots, (J_n^l)$ be the sequences of order intervals associated with $\mathcal{U}$, respectively $\mathcal{V}$, through Definition 2.2. Set

$$\Psi = \{(i,j): [\{I_n^i \times I_n^i : n \in \mathbb{N}\}] \circ [\{J_n^j \times J_n^j : n \in \mathbb{N}\}] \text{ exists } \}$$

Then, according to [7, Lemma 2.1.1 (i)], it follows that

$$\mathcal{U} \circ \mathcal{V} \supseteq \bigcap \quad \{[\{I_n^i \times I_n^i : n \in \mathbb{N}\}] \circ [\{J_n^j \times J_n^j : n \in \mathbb{N}\}] : (i,j) \in \Psi\} \tag{2.21}$$

According to [7, Lemma 2.1.1 (ii)] the inclusion (2.21) implies

$$\mathcal{U} \circ \mathcal{V} \supseteq \bigcap \quad \{[\{J_n^j \times I_n^i : n \in \mathbb{N}\}] : (i,j) \in \Psi\} \tag{2.22}$$

It follows from [7, Lemma 2.1.1 (iii)] $(i,j) \in \Psi$ if and only if

$$\begin{aligned} \forall \quad & m, n \in \mathbb{N}: \\ & I_n^i \cap J_m^j \neq \emptyset \end{aligned} \tag{2.23}$$

Since $\mathcal{ML}_0^0(\Omega)$ is a lattice by Lemma 2.4, it follows that the sequences $\left(\lambda_n^{(i,j)}\right)$ and $\left(\mu_n^{(i,j)}\right)$, defined through

$$\lambda_n^{(i,j)} = \inf\{\lambda_n^i, \lambda_n^j\}, \mu_n^{(i,j)} = \sup\{\mu_n^i, \mu_n^j\} \tag{2.24}$$

for all $(i,j) \in \Psi$, are well defined in $\mathcal{ML}_0^0(\Omega)$. Here $(\lambda_n^i), (\lambda_n^j), (\mu_n^i)$ and (μ_n^j) are the sequences defining the order intervals I_n^i and J_n^j, respectively, through (2.20). For $(i,j) \in \Psi$ and $n \in \mathbb{N}$, set $I_n^{(i,j)} = \left[\lambda_n^{(i,j)}, \mu_n^{(i,j)}\right]$. In view of (2.24), it follows that

$$\begin{aligned} \forall \quad & (i,j) \in \Psi, n \in \mathbb{N}: \\ & I_n^{(i,j)} \supseteq I_n^i \text{ and } I_n^{(i,j)} \supseteq J_n^j \end{aligned}$$

Therefore (2.22) implies that

$$\mathcal{U} \circ \mathcal{V} \supseteq \bigcap \quad \left\{\left[\left\{I_n^{(i,j)} \times I_n^{(i,j)} : n \in \mathbb{N}\right\}\right] : (i,j) \in \Psi\right\} \tag{2.25}$$

Since $\mathcal{NL}(\Omega)$ is σ-distributive by [20, Proposition 4], it follows by (2.20) and (2.24) that

$$\begin{aligned} & \forall (i,j) \in \Psi: \\ & 1) \ \sup\left\{\lambda_n^{(i,j)} : n \in \mathbb{N}\right\} = \sup\{u_i, u_j\} (2.26) \\ & 2) \ \inf\left\{\mu_n^{(i,j)} : n \in \mathbb{N}\right\} = \inf\{u_i, u_j\} \quad (2.26) \end{aligned}$$

where u_i and u_j are the elements of $\mathcal{NL}(\Omega)$ associated with the sequences (I_n^i) and (J_n^j), respectively, of order intervals through (2.20). Since $I_n^i \cap J_n^j \neq \emptyset$ or each $(i,j) \in \Psi$ and $n \in \mathbb{N}$, it follows that

$$\begin{aligned} \forall \quad & (i,j) \in \Psi, n \in \mathbb{N}: \\ & \lambda_n^i \leq \mu_n^j, \lambda_n^j \leq \mu_n^j \end{aligned} \tag{2.27}$$

It follows from (2.20) and (2.27) that

$$u_i \leq u_j$$

and

$$u_j \leq u_i$$

for all $(i,j) \in \Psi$. This shows that $u_i = u_j$ whenever $(i,j) \in \Psi$, so that it follows from (2.26) that

$$\begin{array}{ll} \forall & (i,j) \in \Psi: \\ \exists & u_{(i,j)} \in \mathcal{NL}(\Omega): \\ & \sup\left\{\lambda_n^{(i,j)}: n \in \mathbb{N}\right\} = u_{(i,j)} = \inf\left\{\mu_n^{(i,j)}: n \in \mathbb{N}\right\} \end{array}$$

Thus it follows that $\mathcal{U} \circ \mathcal{V}$ belongs to $\mathcal{J}_1$.
We now show that $\mathcal{J}_1$ is Hausdorff. In this regard, consider functions $u, v \in \mathcal{ML}_0^0(\Omega)$, and a filter $\mathcal{U} \in \mathcal{J}_1$. For the sake of obtaining a contradiction, we assume that

$$(u,v) \in U, U \in \mathcal{U}. \tag{2.28}$$

Let $(\lambda_n^1), \ldots, (\lambda_n^k)$ be the decreasing sequences, and $(\mu_n^1), \ldots, (\mu_n^k)$ the increasing sequences associated with $\mathcal{U}$ through (2.20). It follows from the relation (2.28) that

$$\begin{array}{ll} \forall & n \in \mathbb{N}: \\ \exists & j_n \in \{1, \ldots, k\}: \\ & u, v \in \left[\lambda_n^{j_n}, \mu_n^{j_n}\right] \end{array} \tag{2.29}$$

Since each of the sequences (λ_n^j) is increasing, while all the sequences (μ_n^j) are decreasing, it follows from (2.29) that

$$\begin{array}{ll} \exists & j \in \{1, \ldots, k\}: \\ \forall & n \in \mathbb{N}: \\ & u, v \in \left[\lambda_n^j, \mu_n^j\right] \end{array} \tag{2.30}$$

Since

$$\sup\{\lambda_n^j: n \in \mathbb{N}\} = u_j = \inf\{\mu_n^j: n \in \mathbb{N}\}$$

for some $u_j \in \mathcal{NL}(\Omega)$, it follows from (2.30) that $u = u_j = v$. It now follows from [7, Proposition 2.1.10] that $\mathcal{J}_1$ is Hausdorff.

The proof of Proposition 2.3 requires the following lemma.
Lemma 2.4. The set $\mathcal{ML}_0^0(\Omega)$ is a σ-distributive lattice with respect to the pointwise order.
Proof. Consider functions $u, v \in \mathcal{ML}_1^0(\Omega)$, and set

$$w = \sup\{u, v\} \in \mathcal{ML}^0(\Omega) \tag{2.31}$$

We show that $w \in \mathcal{ML}_1^0(\Omega)$. According to (??) it follows from (2.31) that

$$w(x,t) = (I \circ S)\varphi(x,t), (x,t) \in \Omega \tag{2.32}$$

where

$$\varphi(x,t) = \sup\{u(x,t), v(x,t)\}. \tag{2.33}$$

Since $u, v \in \mathcal{ML}_0^0(\Omega)$, it follows from (2.2) and (2.33) that $\varphi(x,0) \in \mathbb{R}$ and that φ is continuous at ($x, 0$) for all $x \in [-a,a]$. Since φ is continuous at each point ($x, 0$), it follows that it is both upper semi-continuous and lower semi-continuous at each such point. Therefore

$$w(x,0) = I(S(\varphi))(x,0) = \varphi(x,0) \in \mathbb{R}, x \in [-a,a]. \tag{2.34}$$

Thus it remains to show that w is continuous at ($x, 0$), for $x \in [-a,a]$. In this regard, we note (2.32) and (??) imply that w is lower semi-continues. Hence we need only show that w is also upper semi-continuous at ($x, 0$). In this regard, consider a fixed $x \in [-a,a]$ and $M \in \bar{R}$ such that

$$w(x,0) = \varphi(x,0) < M \tag{2.35}$$

Since φ is continuous, and hence upper semi-continuous at ($x, 0$), it follows from (2.35) that

$$\begin{aligned} \exists \quad & V \in \mathcal{V}(x): \\ \forall \quad & (y,t) \in V \cap \Omega: \\ & \varphi(y,t) < M \end{aligned} \tag{2.36}$$

Since $V \cap \Omega$ is open in Ω, the relation (2.36), together with the definition (??) of S imply that

$$\begin{aligned} \forall \quad & (y,t) \in V \cap \Omega: \\ & S(\varphi)(y,t) < M \end{aligned} \tag{2.37}$$

Finally, it follows from (2.37) and (??) that

$$\begin{aligned} \forall \quad & (y,t) \in V \cap \Omega: \\ & w(y,t) = I(S(\varphi))(y,t) < M \end{aligned}$$

so that w is upper semi-continuous at ($x, 0$). Since $x \in [-a,a]$ is arbitrary, it follows that $w \in \mathcal{ML}_0^0(\Omega)$.
In the same way, it follows that $\inf\{u, v\}$ belongs to $\mathcal{ML}_0^0(\Omega)$. This shows that $\mathcal{L}_0^0(\Omega)$ is a sublattice of $\mathcal{ML}^0(\Omega)$. Since $\mathcal{ML}^0(\Omega)$ is σ-distributive according to [20, Corollary 5], it follows that $\mathcal{ML}_0^0(\Omega)$ is also σ-distributive. This completes the proof.

Since $\mathcal{ML}_0^0(\Omega)$ is a Hausdorff uniform convergence space with respect to $\mathcal{J}_1$, we may construct a completion of $\mathcal{ML}_0^0(\Omega)$. In particular, see [23], there exists a complete Hausdorff uniform convergence space $\mathcal{ML}_0^0(\Omega)^\sharp$, and a uniformly continuous embedding

$$\iota: \mathcal{ML}_0^0(\Omega) \to \mathcal{ML}_0^0(\Omega)^\sharp$$

such that $\iota(\mathcal{ML}_0^0(\Omega))$ is dense in $\mathcal{ML}_0^0(\Omega)^\#$, with the following universal property: If Y is a complete, Hausdorff uniform convergence space and

$$T: \mathcal{ML}_0^0(\Omega) \to Y$$

is uniformly continuous, then there exists a uniformly continuous mapping

$$T^{\sharp}: \mathcal{ML}_0^0(\Omega)^{\sharp} \to Y$$

such that $T = T^{\sharp} \circ \iota$.
We now proceed to obtain a concrete characterization of the space $\mathcal{ML}_0^0(\Omega)^{\sharp}$. In this regard, we characterize the Cauchy filters with respect to $\mathcal{J}_1$.

Proposition 2.5. A filter $\mathcal{F}$ on $\mathcal{ML}_0^0(\Omega)$ is a Cauchy filter with respect to $\mathcal{J}_1$ if and only if

$$\exists (\lambda_n), (\mu_n) \subset \mathcal{ML}_0^0(\Omega):$$
$$\exists u \in \mathcal{NL}(\Omega): (2.38)$$

1. $\lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n$
2. $\lambda_n(x,0) = \mu_n(x,0), x \in [-a,a], n \in \mathbb{N}$
3. $\sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\}$
4. $[\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F}$

Proof. Let (2.38) hold. Then set $\lambda_n^1 = \lambda_n, \mu_n^1 = \mu_n \subset \mathcal{ML}_0^0(\Omega)$ and $u_1 = u \in \mathcal{NL}(\Omega)$. Therefore λ_n^1, μ_n^1 and u_1 satisfy (2.20)(1) - (3). Furthermore, (2.38)(4) implies for each $n \in \mathbb{N}$ there exists a set $A \in \mathcal{F}$ such that $A \subseteq [\lambda_n^1, \mu_n^1]$. This implies $A \times A \subseteq [\lambda_n^1, \mu_n^1] \times [\lambda_n^1, \mu_n^1]$, which implies that

$$[\{[\lambda_n^1, \mu_n^1] \times [\lambda_n^1, \mu_n^1] : n \in \mathbb{N}\}] \subseteq \mathcal{F} \times \mathcal{F}$$

Therefore by Definition 2.2$\mathcal{F} \times \mathcal{F} \in \mathcal{J}_1$ for $k = 1$. Thus $\mathcal{F}$ is a Cauchy filter with respect to $\mathcal{J}_1$.
Conversely, Let$\mathcal{F}$ be a Cauchy filter on $\mathcal{ML}_0^0(\Omega)$ so that $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_1$. Let $(\lambda_n^1), \ldots, (\lambda_n^k)$ denote the decreasing sequences, and $(\mu_n^1), \ldots, (\mu_n^k)$ the increasing sequences associated with $\mathcal{F} \times \mathcal{F}$ through Definition 2.2. Let

$$\lambda_n = \inf\{\lambda_n^1, \cdots \lambda_n^k\} \tag{2.39}$$

and

$$\mu_n = \sup\{\mu_n^1, \cdots, \mu_n^k\} \tag{2.40}$$

Then λ_n is an are increasing sequence and μ_n is a decreasing sequence. It follows that (2.38)(1) - (2) hold. For fix $k \in \mathbb{N}$,

The following is an immediate consequence of Proposition 2.5
Corollary 2.6. A filter $\mathcal{F}$ on $\mathcal{ML}_0^0(\Omega)$ converges to $u \in \mathcal{ML}_0^0(\Omega)$ with respect to the convergence structure induced by $\mathcal{J}_1$ if and only if

$$\exists (\lambda_n), (\mu_n) \subset \mathcal{ML}_0^0(\Omega):$$
1) $\lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n$
2) $\lambda_n(x,0) = u(x,0) = \mu_n(x,0), x \in [-a,a], n \in \mathbb{N}$
3) $\sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\}$
4) $[\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F}$

We denote the convergence structure induced by $\mathcal{J}_1$ on $\mathcal{ML}_0^0(\Omega)$ by λ_1.
The Wyler completion of $\mathcal{ML}_0^0(\Omega)$ is constructed in the following way. Denote by $C[\mathcal{ML}_0^0(\Omega)]$ the set of all Cauchy filters on $\mathcal{ML}_0^0(\Omega)$, and define an equivalence relation on $C[\mathcal{ML}_0^0(\Omega)]$ through

$$\mathcal{F} \sim_C \mathcal{G} \Leftrightarrow \mathcal{F} \cap \mathcal{G} \in C[\mathcal{ML}_0^0(\Omega)]. \tag{2.41}$$

We denote by $\mathcal{ML}_0^0(\Omega)^\#$ the quotient space $C[\mathcal{ML}_0^0(\Omega)]/{\sim_C}$. For $\mathcal{F} \in C[\mathcal{ML}_0^0(\Omega)]$, the equivalence class generated by $\mathcal{F}$ with respect to (2.41) is denoted $[\mathcal{F}]$. One may identify $\mathcal{ML}_0^0(\Omega)$ with a subset of $\mathcal{ML}_0^0(\Omega)^\#$ by associating each $u \in \mathcal{ML}_0^0(\Omega)$ with $\lambda_1(u) \subset C[\mathcal{ML}_0^0(\Omega)]$. It is an immediate consequence of the definition of a convergence structure [7, Definition 1.1.1] that $\lambda_1(u)$ is indeed a $\sim_C$-equivalence class. Furthermore, since λ_1 is Hausdorff, the mapping

$$\mathcal{ML}_0^0(\Omega) \ni u \mapsto \lambda_1(u) \in \mathcal{ML}_0^0(\Omega)^\sharp$$

is injective. Thus we may indeed consider $\mathcal{ML}_0^0(\Omega)$ as a subset of $\mathcal{ML}_0^0(\Omega)^\sharp$

The Wyler completion of $\mathcal{ML}_0^0(\Omega)$ is the set $\mathcal{ML}_0^0(\Omega)^\#$, equipped with the following uniform convergence structure, see for instance [17]:

$$\mathcal{U} \in \mathcal{J}^\sharp \Leftrightarrow \begin{pmatrix} \exists & n \in \mathbb{N}: \\ \exists & \mathcal{U}_1, \dots, \mathcal{U}_n \in \mathcal{J}_1 \\ & [\mathcal{U}_1] \cap \dots \cap [\mathcal{U}_n] \subseteq \mathcal{U} \end{pmatrix} \tag{2.42}$$

Note that each $\mathcal{U}_i$, for $i = 1, \dots, k$, is a filter on $\mathcal{ML}_0^0(\Omega) \times \mathcal{ML}_0^0(\Omega)$, and $[\mathcal{U}_i]$ denotes the filter generated by $\mathcal{U}_i$ in $\mathcal{ML}_0^0(\Omega)^\# \times \mathcal{ML}_0^0(\Omega)^\#$.

We now give a concrete description of the completion $\mathcal{ML}_0^0(\Omega)^\sharp$ of $\mathcal{ML}_0^0(\Omega)$ as a subset of $\mathcal{NL}(\Omega)$. In this regard, we introduce the space

$$\mathcal{NL}_0(\Omega) = \left\{ u \in \mathcal{NL}(\Omega) \,\middle|\, \begin{array}{ll} \exists & \lambda, \mu \in \mathcal{ML}_0^0(\Omega) \\ & 1)\ \lambda \le u \le \mu \\ & 2)\ \lambda(x,0) = \mu(x,0), x \in [-a,a] \end{array} \right\} \tag{2.43}$$

Proposition 2.7. For every $\mathcal{F} \in C[\mathcal{ML}_0^0(\Omega)]$, let $u_\mathcal{F}$ denote the element of $\mathcal{NL}(\Omega)$ associated with $\mathcal{F}$ through (2.38). Then the mapping

$$C[\mathcal{ML}_0^0(\Omega)] \ni [\mathcal{F}] \mapsto u_\mathcal{F} \in \mathcal{NL}_0(\Omega) \tag{2.44}$$

is a bijection.

Proof. It is clear from (2.38) and (2.43) that $u_\mathcal{F} \in \mathcal{NL}_0$ whenever $\mathcal{F} \in C[\mathcal{ML}_0^0(\Omega)]$. We now claim that

$$\forall \mathcal{F}, \mathcal{G} \in C[\mathcal{ML}_0^0(\Omega)]: \tag{2.45}$$
$$\mathcal{F} \sim_C \mathcal{G} \Rightarrow u_\mathcal{F} = u_\mathcal{G} \tag{2.45}$$

In this regard, consider $\mathcal{F}, \mathcal{G} \in C[\mathcal{ML}_0^0(\Omega)]$ such that $\mathcal{F} \sim_C \mathcal{G}$. According to (2.41) the filter $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ is a Cauchy filter. Let (λ_n^0) and (μ_n^0) be the sequences associated with $\mathcal{F}$, and (λ_n^1) and (μ_n^1) the sequences associated with $\mathcal{H}$, through Proposition 2.5. Since $\mathcal{H} \subseteq \mathcal{F}$, it follows from (2.38) that

$$\begin{array}{ll} \forall & n \in \mathbb{N}: \\ & [\lambda_n^0, \mu_n^0] \cap [\lambda_n^1, \mu_n^1] \neq \emptyset \end{array}$$

so that

$$\begin{array}{c} \forall\, n \in \mathbb{N}: \\ \exists\, u \in \mathcal{ML}_0^0(\Omega): \\ 1)\ \lambda_n^0 \leq u \leq \mu_n^0 (2.46) \\ 2)\ \lambda_n^1 \leq u \leq \mu_n^1 (2.46) \end{array}$$

It follows from (2.46) that

$$\begin{array}{c} \forall n \in \mathbb{N}: \\ 1)\ \lambda_n^0 \leq \mu_n^1 (2.47) \\ 2)\ \lambda_n^1 \leq \mu_n^0 (2.47) \end{array}$$

Since (λ_n^0) increases to $u_{\mathcal{F}}$, while (μ_n^1) decreases to $u_{\mathcal{H}}$, it follows from (2.47) that $u_{\mathcal{F}} \leq u_{\mathcal{H}}$. Similarly, we find that $u_{\mathcal{H}} \leq u_{\mathcal{F}}$, thus $u_{\mathcal{F}} = u_{\mathcal{H}}$. In exactly the same way we may verify that $u_{\mathcal{G}} = u_{\mathcal{H}}$. Hence $u_{\mathcal{F}} = u_{\mathcal{G}}$, which verifies (2.45). Thus the mapping (2.44) is well defined.

We now show that the mapping (2.44) is injective. In this regard, consider $\mathcal{F}, \mathcal{G} \in C[\mathcal{ML}_0^0(\Omega)]$ such that $u_{\mathcal{F}} = u_{\mathcal{G}} = u$. Let (λ_n^0) and (μ_n^0) be the sequences associated with $\mathcal{F}$, and (λ_n^1) and (μ_n^1) the sequences associated with $\mathcal{G}$, through Proposition 2.5. In view of (2.38), it follows that

$$u = \sup\{\lambda_n^0 : n \in \mathbb{N}\} = \inf\{\mu_n^0 : n \in \mathbb{N}\} = \sup\{\lambda_n^1 : n \in \mathbb{N}\} = \inf\{\mu_n^1 : n \in \mathbb{N}\} \qquad (2.48)$$

Since $\mathcal{NL}(\Omega)$ is σ-distributive, see [20, Proposition 3], it follows from (2.48) that the sequence (λ_n) defined through

$$\lambda_n = \inf\{\lambda_n^0, \lambda_n^1\}$$

increase to u, while the sequence (μ_n) defined through

$$\mu_n = \sup\{\mu_n^0, \mu_n^1\}$$

decreases to u. Furthermore, since

$$[\lambda_n^0, \mu_n^0] \subseteq [\lambda_n, \mu_n] \text{ and } [\lambda_n^1, \mu_n^1] \subseteq [\lambda_n, \mu_n]$$

it follows that

$$[\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F} \cap \mathcal{G}. \qquad (2.49)$$

Since $\mathcal{ML}_0^0(\Omega)$ is a lattice by Lemma 2.4, it follows that $(\lambda_n), (\mu_n) \subseteq \mathcal{ML}_0^0(\Omega)$. Therefore (2.49) implies that $\mathcal{F} \cap \mathcal{G} \in C[\mathcal{ML}_0^0(\Omega)]$. Hence $\mathcal{F} \sim_C \mathcal{G}$, which verifies that the mapping (2.44) is injective.

We now show that the mapping (2.44) is surjective. To see that this is so, let $u \in \mathcal{NL}_0(\Omega)$ be arbitrary but fixed. According to [20, Theorem 14] there exists an increasing sequence $(v_n) \subseteq \mathcal{ML}^0(\Omega)$ with the property that

$$u = \sup\{v_n : n \in \mathbb{N}\} \qquad (2.50)$$

Using the same techniques employed in the proof of [20, Theorem 14], we may show that there is a decreasing sequence $(u_n) \subseteq \mathcal{ML}^0(\Omega)$ such that

$$u = \inf\{u_n : n \in \mathbb{N}\} \qquad (2.51)$$

Define the sequences (λ_n) and (μ_n) in $\mathcal{ML}^0(\Omega)$ by

$$\lambda_n = \sup\{v_n, \lambda\}, \mu_n = \inf\{u_n, \mu\} \tag{2.52}$$

Since $\mathcal{NL}(\Omega)$ is σ-distributive, see [20, Proposition 3], and $\lambda \leq u \leq \mu$, it follows from (2.51) and (2.52) that

$$\sup\{\lambda_n : n \in \mathbb{N}\} = \sup\{u, \lambda\} = u = \inf\{u, \mu\} = \inf\{\mu_n : n \in \mathbb{N}\} \tag{2.53}$$

Furthermore, since

$$\lambda \leq \lambda_n \leq \mu_n \leq \mu, n \in \mathbb{N}$$

it follows from (2.43) that

$$\begin{array}{ll} \forall & n \in \mathbb{N}, x \in [-a, a]: \\ & \lambda_n(x, 0) = \mu_n(x, 0) = u(x, 0) \in \mathbb{R} \end{array}$$

Moreover, an easy computation verifies that that (2.53) implies the continuity of λ_n and μ_n at $(x, 0)$ for all $x \in [-a, a]$. Thus $u = u_{\mathcal{F}}$, where $\mathcal{F}$ is the filter

$$\mathcal{F} = [\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}].$$

This shows that the mapping (2.44) is a surjection. Thus the proof is complete.
In view of Proposition 2.7, we can consider $\mathcal{NL}_0(\Omega)$ as the completion of $\mathcal{ML}_0^0(\Omega)$ with respect to the uniform convergence structure $\mathcal{J}_1^{\#}$. In particular, the uniform convergence structure is the initial uniform convergence structure with respect to the mapping (2.44) and the uniform convergence structure (2.42) on $C[\mathcal{M}_0^0(\Omega)]$. An explicit description of the uniform convergence structure $\mathcal{J}_1^{\sharp}$ on $\mathcal{NL}_0(\Omega)$ is given by the following: A filter $\mathcal{U}$ on $\mathcal{NL}_0(\Omega) \times \mathcal{NL}_0(\Omega)$ belongs to $\mathcal{J}_1^{\sharp}$ if and only if

$$\begin{array}{l} \exists k \in \mathbb{N}: \\ \forall j = 1, \ldots, k: \\ \exists (\lambda_n^j), (\mu_n^j) \subseteq \mathcal{NL}_0(\Omega): \\ \exists u_j \in \mathcal{NL}(\Omega): \\ \ 1)\ \lambda_n^j \leq \lambda_{n+1}^j \leq \mu_{n+1}^j \leq \mu_n^j, n \in \mathbb{N} \\ \ 2)\ \lambda_n^j(x, 0) = \mu_n^j(x, 0), x \in [-a, a], n \in \mathbb{N} \\ \ 3)\ \sup\{\lambda_n^j : n \in \mathbb{N}\} = u_j = \inf\{\mu_n^j : n \in \mathbb{N}\} \\ \ 4)\ [\{\bar{I}_n^1 \times \bar{I}_n^1 : n \in \mathbb{N}\}] \cap \ldots \cap [\{\bar{I}_n^k \times \bar{I}_n^k : n \in \mathbb{N}\}] \end{array}$$

Here $\bar{I}_n^j$ denotes the set

$$\bar{I}_n^j = \{v \in \mathcal{ML}_0^0(\Omega) : \lambda_n^j \leq v \leq \mu_n^j\}$$

The space $\mathcal{ML}_0^1(\Omega)$ is equipped with the initial uniform convergence structure $\mathcal{J}_D$ with respect to the mappings (2.3), (2.6) and the inclusion (2.5), where $\mathcal{ML}^0(\Omega)$ carries the uniform order convergence structure, see Definition ??. That is, for a filter $\mathcal{U}$ on $\mathcal{ML}_0^1(\Omega)$ we have

$$\mathcal{U} \in \mathcal{J}_D \Leftrightarrow \begin{pmatrix} 1) & \mathcal{U} \in \mathcal{J}_1 \\ 2) & (\mathcal{D}_x \times \mathcal{D}_x)(\mathcal{U}) \in \mathcal{J}_1 \\ 3) & (\mathcal{D}_t \times \mathcal{D}_t)(\mathcal{U}) \in \mathcal{J}_0 \end{pmatrix} \tag{2.54}$$

In particular, a filter $\mathcal{F}$ on $\mathcal{ML}_0^1(\Omega)$ converges to $u \in \mathcal{ML}_0^1(\Omega)$ if and only if $\mathcal{F}$ and $\mathcal{D}_x(\mathcal{F})$ converge to u and $\mathcal{D}_x u$, respectively, in $\mathcal{ML}_0^1(\Omega)$, while $\mathcal{D}_t(\mathcal{F})$ converges to $\mathcal{D}_t u$ in $\mathcal{ML}^0(\Omega)$.

An application of [22, Theorem 4.3] shows that the completion of $\mathcal{ML}_0^1(\Omega)$, which we denote by $\mathcal{NL}_0^1(\Omega)$, may be represented as a subset of

$$\mathcal{NL}_1(\Omega) \times \mathcal{NL}_1(\Omega) \times \mathcal{NL}(\Omega).$$

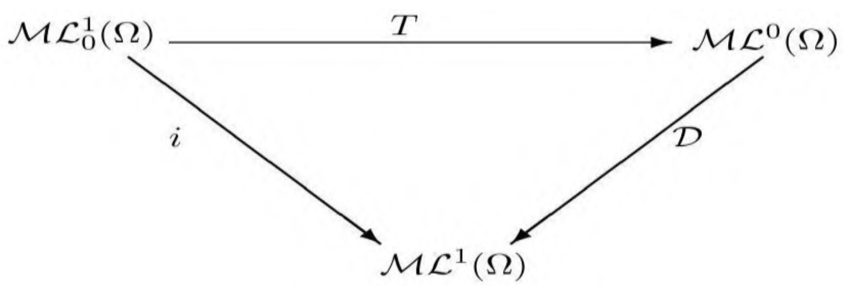
In particular, the mapping

$$\mathbf{D}^\sharp : \mathcal{NL}_0^1(\Omega) \ni u \mapsto \left(u, D_x^\sharp u, D_t^\sharp u\right) \in \mathcal{NL}_1(\Omega) \times \mathcal{NL}_1(\Omega) \times \mathcal{NL}(\Omega) \tag{2.55}$$

is injective and uniformly continuous. Here $\mathcal{D}_x^\sharp$ and $\mathcal{D}_t^\sharp$ are the uniformly continuous extensions of the mappings (2.4) and (2.6).

In order to extend to Cauchy problem (1.1-1.2) to generalized functions in $\mathcal{NL}_0^1(\Omega)$, we have to extend the mapping (2.9) to $\mathcal{NL}_0^1(\Omega)$ in a meaningful way. In this regard, the following is the basic results.

Theorem 2.8. The mapping T is uniformly continuous.
Proof. This follows from commutative diagram



since the inclusion mapping

$$i : \mathcal{ML}_{u_0}^1(\Omega) \longrightarrow \mathcal{ML}^0(\Omega)$$

is uniformly continuous and the nonlinear PDE operator $\mathcal{D}$ is uniformly continuous.
Theorem 2.9. The mapping $\bar{T} : \mathcal{ML}_0^1(\Omega) \to \mathcal{ML}^0(\Omega) \times C^1[-a, a]$ is uniformly continuous. The proof of this result is rather lengthy and technical. Furthermore, it follows essentially the same methods used in the proof of [21, Theorem 6]. Thus we omit it.

In view of Theorem 2.9, there exists a unique uniformly continuous mapping

$$\bar{T}^\sharp : \mathcal{NL}_0^1(\Omega) \to \mathcal{NL}(\Omega) \times C^1[-a, a] \tag{2.57}$$

which extends the mapping (2.9). Any solution $u \in \mathcal{NL}_0^1(\Omega)$ of the equation

$$\bar{T}^{\sharp}u = (f, u_0) \tag{2.58}$$

is therefore considered a generalized solution of the initial value problem (1.1-1.2).

3 Existence of Generalized Solutions

In this section we give our main results of existence of generalized solution to the Cauchy problem (1.1-1.2).

3.1 Existence result for Pull-back spaces

We state the approximation result which we shall use in sequel to prove existence and uniqueness of generalized solution to the Cauchy problem (1.1-1.2), see [19] and [21].
Theorem 3.1. Let $f \in C^0(\bar{\Omega})$. Then

$$\begin{array}{ll} & \forall \epsilon > 0: \\ & \exists \delta > 0: \\ & \forall (x_0, t_0) \in \Omega: \\ \exists u = u_{\epsilon, x_0, t_0} \in C^1(\Omega): & (3.1) \\ \forall (x,t) \in \Omega: & (3.1) \\ \begin{pmatrix} |x - x_0| < \delta \\ |t - t_0| < \delta \end{pmatrix} \Rightarrow f(x,t) - \epsilon < Tu(x,t) \leq f(x,t) & (3.1) \end{array}$$

Furthermore,

$$\begin{array}{ll} & \forall \epsilon > 0: \\ & \exists \delta > 0: \\ & \forall x_0 \in [-a, a]: \\ \exists u = u_{\epsilon, x_0} \in C^1(\Omega): & (3.2) \\ (a) \forall (x,t) \in \Omega: & (3.2) \\ \begin{pmatrix} |x - x_0| < \delta \\ |t| < \delta \end{pmatrix} \Rightarrow f(x,t) - \epsilon < Tu(x,t) \leq f(x,t) & (3.2) \\ \text{(b) } u(x,0) = u_0(x), |x - x_0| < \delta & (3.2) \end{array}$$

Proof. The proof is similar to that of [16, Lemma 8.1]. See also, [19] and [21]. In order to proof existence and uniqueness of solution using the above approximation result, we introduce a finite initial adaptive δ-tiling in Ω. Given any $\delta > 0$ one can always find at least one initial adaptive δ-tiling in Ω, see [16, Section 8]. A finite initial adaptive δ-tiling in Ω is a subset of Ω such that for any $\delta > 0$ there is a finite collection $\mathcal{K}_\delta = \{K_i\}, i = 1 \cdots m, m \in \mathcal{N}$ of perfect, compact subsets of $\mathbb{R}^2$ with pairwise disjoint interiors such that

$$\begin{array}{l} \forall K_i \in \mathcal{K}_\delta \\ \quad (x,t), (x_0, t_0) \in K_i \Rightarrow \begin{pmatrix} |x - x_0| < \delta \\ |t - t_0| < \delta \end{pmatrix} \end{array}$$

and

$$\{(x,0): -a \le x \le a\} \bigcap \left(\bigcup_{K_i \in \mathcal{K}_\delta} \partial K_i \right) \text{ is finite.}$$

Our main result for this section is the following.

Theorem 3.2. If $u_0 \in C^1[-a,a]$ then there exist a unique solution $u^\# \in \mathcal{NL}^1_{0,\bar{T}}(\Omega)$ of the Cauchy problem (1.1-1.2) that satisfies (2.19).

Proof. For any $n \in \mathbb{N}$, set $\epsilon_n = \frac{1}{n}$. Then applying Lemma3.1, we have that

$$\begin{aligned} &\forall (x_0,t_0) \in \Omega: \\ &\exists \delta_n > 0: \\ &\exists u = u_{n,x_0,t_0} \in C^1(\Omega): \quad (3.3) \\ &\forall (x,t) \in \Omega: \quad (3.3) \\ &\begin{pmatrix} |x-x_0| < \delta \\ |t-t_0| < \delta \end{pmatrix} \Rightarrow f(x,t) - \frac{1}{2n} < Tu(x,t) \le f(x,t) (3.3) \end{aligned}$$

and

$$\begin{aligned} &\forall x_0 \in [-a,a]: \\ &\exists u = u_{n,x_0} \in C^1(\Omega): \\ &(a) \forall (x,t) \in \Omega: \quad (3.4) \\ &\begin{pmatrix} |x-x_0| < \delta \\ |t| < \delta \end{pmatrix} \Rightarrow f(x,t) - \frac{1}{2n} < Tu(x,t) \le f(x,t) (3.4) \\ &(b)\ u(x,0) = u_0(x), x \in [-a,a], |x-x_0| < \delta \quad (3.4) \end{aligned}$$

Let $\mathcal{K}_{\delta_n}$ be a finite adaptive δ_n-tiling. For $K_i \in \mathcal{K}_{\delta_n}$, set

$$u_n^i = \begin{cases} u, if \left(\begin{matrix} \text{for any } (x_0,y_0) \in \text{int}(K_i) \\ \text{int}(K_i) \bigcap \{(x,0): x \in [-a,a]\} = \emptyset \end{matrix} \right. \\ u, if \left(\begin{matrix} \text{for any } (x_0,0) \in \text{int}(K_i) \bigcap ([-a,a] \cap \{0\}) \\ \text{int}(K_i) \bigcap \{(x,0): x \in [-a,a]\} \neq \emptyset \end{matrix} \right. \end{cases}$$

Now we define $u_n \in \mathcal{ML}^1_0(\Omega)$ as

$$u_n = I \circ S \left(\sum_{i=1}^{m} \chi_i u_n^i \right),$$

where χ_i is the characteristic function defined with respect to the set $\text{int}(K_i)$. Then from (3.3), (3.4) we have that

$$f(x,t) - \frac{1}{n} < Tu_n(x,t) \le f(x,t) \quad (3.5)$$

and

$$u_n(x,0) = u_0(x), x \in [-a,a] \quad (3.6)$$

Let $U_n \in \mathcal{ML}^1_{0,\bar{T}}(\Omega)$ be the $\sim_{\bar{T}}$ equivalence class generated by u_n. Then by (2.11), (2.12), (3.5) and (3.6), we see that the sequence $\left(\hat{T}U_n\right)$ converges to (f, u_0) which implies that $\left(\hat{T}U_n\right)$ is a Cauchy sequence in $\mathcal{ML}^0(\Omega) \times C^1[-a, a]$. It follows, from the fact that $\hat{T}$ is uniformly continuous, that the sequence U_n is a Cauchy sequence in $\mathcal{ML}^1_{0,\bar{T}}$. This means that U_n converges to a point $u^\sharp \in \mathcal{NL}^1_{0,\bar{T}}$ such that $\hat{T}^\sharp u^\sharp = (f, u_0)$. This proves the existence part. Since the mapping $\hat{T}^\sharp$ is injective, it follows that $u^\sharp \in \mathcal{NL}1_{0,\bar{T}}$ is Unique.

3.2 Existence result for Sobolev-type spaces

In this section we give the main existence result for Sobolev-type spaces.
Theorem 3.3. Let $u_0 \in C[-a, a]$. Then there exists $u^\sharp \in \mathcal{NL}^1_0(\Omega)$, a generalized solution of (1.1 - 1.2) such that

$$\bar{T}^\sharp u^\sharp = (f, u_0).$$

Proof. Let $\mathcal{K}_\delta$ be a finite adaptive δ - tiling in Ω. Let $K_i \in \mathcal{K}_\delta$ be a fixed and arbitrary. Assume that for each $i \in \mathbb{N}, \mathcal{S} \cap K_i = \emptyset$, or $\mathcal{S} \cap K_i \neq \emptyset$, where $\mathcal{S}$ is the non-characteristic hyperplane

$$\mathcal{S} = \{(x, 0) : x \in (-a, a)\}$$

Then from the continuity of the function f and the fact that the mapping F is open and surjective [21], we have that

$\forall (x_0, t_0) \in K_i$

$\exists \xi(x_0, t_0) \in \mathbb{R}^4, F\left(x_0, t_0, \xi(x_0, t_0)\right) = f(x_0, t_0)$:

$\exists \delta = \delta_{x_0,t_0}, \varepsilon > 0$:

1)$\{(x, t, f(x, t)) ⊢|(|x - x\_0 | < \delta, @|t - t\_0 | < \delta)⊣ \} \subseteq \text{int}\{(x, t, F(x, t, \xi)) ⊢|(|x - x\_0 | < \delta|t - t\_0 | < \delta, @|\xi(x, t) - \xi(x\_0, t\_0)| < \varepsilon)⊣ \}$(3.7)

2)$F: B\_\delta\ (x\_0, t\_0\ ) \times B\_2\varepsilon\ (\xi(x\_0, t\_0\ )) \longrightarrow \mathbb{R}"\ open.$

In particular, if we set $t_0 = 0$ we can take

$$\xi(x_0, 0) = \frac{d}{dx}\xi(x_0) \tag{3.8}$$

For each point $(x_0, t_0) \in K_i$, fix $\xi(x_0, 0) \in \mathbb{R}^4$ in (3.7) so that (3.8) holds at $t_0 = 0$. Since K_i is compact, it follows from (3.7) that

$$\begin{aligned}&\exists\delta>0:\\&\forall(x_0,t_0)\in K_i\\&\exists\varepsilon=\varepsilon_{x_0,t_0}>0:\end{aligned}$$

$$1)\left\{(x,t,f(x,t))\,\middle|\,\begin{matrix}|x-x_0|<\delta\\|t-t_0|<\delta\end{matrix}\right\}\subseteq \operatorname{int}\left\{(x,t,F(x,t,\xi))\,\middle|\,\begin{matrix}|x-x_0|<\delta\\|t-t_0|<\delta\\|\xi(x,t)-\xi(x_0,t_0)|<\varepsilon\end{matrix}\right\} \quad (3.9)$$

$$2)F: B_\delta(x_0,t_0)\times B_{2\varepsilon}\big(\xi(x_0,t_0)\big)\longrightarrow\mathbb{R}\text{ open.} \quad (3.9)$$

Now let $\mathcal{K}_{\delta_1}=\{I_{i_k}\}, k=1\cdots i$ be a finite adaptive δ-tiling of the set K_i such that $\delta_1<\delta$ and for each $k=1,\cdots n$

$$I_{i_k}\cap\mathcal{S}=\emptyset \quad (3.10)$$

or

$$\operatorname{int}I_{i_k}\cap\mathcal{S}\neq\emptyset \quad (3.11)$$

If a_{i_k} with $k=1\cdots n$ is the center of the interval I_{i_k} satisfying (3.10) then by (3.9) we haven that

$$\exists\varepsilon_{i_k}>0:$$

$$1)\left\{(x,t,f(x,t))\mid(x,t)\in I_{i_k}\right\}\subseteq\operatorname{int}\left\{(x,t,F(x,t,\xi))\quad\begin{matrix}(x,t)\in I_{i_k}\\|\xi(x,t)-\xi(a_{i_k})|<\varepsilon_{i_k}\end{matrix}\right\} \quad (3.12)$$

$$2)\ F: I_{i_k}\times B_{2\varepsilon_{i_k}}\left(\xi(a_{i_k})\right)\longrightarrow\mathbb{R}\text{ open.} \quad (3.12)$$

If on the other hand, I_{i_k} satisfies (3.11) then we set a_{i_k} equal to the midpoint of $I_{i_k}\cap\mathcal{S}$. We then obtain (3.12) by (3.9) such that (3.8) holds.

Let $0<\gamma<1$ be fixed. Then by Lemma 3.1 and (3.12) we have

$$\begin{aligned}&\forall(x_0,t_0)\in I_{i_k}\\&\exists U=U_{x_0,t_0}\in C^1(\Omega):\\&(3.13)\end{aligned}$$

$$\exists\delta=\delta_{x_0,t_0}>0:$$

$$(x,t)\in B_{\delta_{x_0}}(x_0,t_0)\cap I_{i_k}\Longrightarrow\begin{cases}1)D_tU(x,t),D_xU(x,t)\in B_{\varepsilon_{i_k}}\left(\xi(a_{i_k})\right)\\2)f(x,t)-\gamma<TU(x,t)<f(x,t)\end{cases} \quad (3.13)$$

Furthermore, if I_{i_k} satisfies (3.11), then we also have that

$$D_xU(x,0)=\frac{d}{dx}u_0(x).$$

Similarly, if we let $\mathcal{K}_{\delta_2}=\left\{J_{i_{k_j}}\right\}, j=1\cdots k$ be the finite adaptive δ-tiling of the set I_{i_k}, such that $\delta_2<\delta_1<\delta$. Then for $j=1,\cdots n$ we have that

$\forall (x,t) \in J_{i_{k_j}}$

$\exists U = U_{i_{k_j}} \in C^1(\Omega)$: (3.14)

1) $D_t U(x,t), D_x U(x,t) \in B_{\varepsilon_{i_{k_j}}}\left(\xi\left(a_{i_{k_j}}\right)\right)$ (3.14)

2) $f(x,t) - \gamma < TU(x,t) < f(x,t)$ (3.14)

and

$$J_{i_{k_j}} \cap \mathcal{S} = \emptyset \tag{3.15}$$

or

$$\mathrm{int} J_{i_{k_j}} \cap \mathcal{S} \neq \emptyset \tag{3.16}$$

If $J_{i_{k_j}}$ satisfies (3.16), then

$$D_x U(x,0) = \frac{d}{dx} u_0(x)$$

Set

$$\Gamma_1 = \Omega \setminus \left(\bigcup_{i \in \mathbb{N}} \left(\bigcup_{k=1}^{i} \left(\bigcup_{j=1}^{k} \mathrm{int} U_{i_{k_j}} \right) \right) \right)$$

and

$$V_1 = \sum_{i \in \mathbb{N}} \left(\sum_{k=1}^{i} \left(\sum_{j=1}^{k} \chi_{i_{k_j}} U_{i_{k_j}} \right) \right)$$

where $\chi_{i_{k_j}}$ is the characteristic function of $J_{i_{k_j}}$. The set Γ_1 is closed and nowhere dense and $V_1 \in C^1(\Omega \setminus \Gamma_1)$. Furthermore, $\Gamma_1 \cap \mathcal{S}$ is closed nowhere dense in $\mathcal{S}$ and

$$D_x V_1(x,0) = \frac{d}{dx} u_0(x) \; \forall (x,0) \in \mathcal{S} \setminus (\Gamma_1 \cap \mathcal{S})$$

From (3.13), we have

$$f(x,t) - \gamma < TV_1(x,t) < f(x,t) \; (x,t) \in \Omega \setminus \Gamma_1.$$

Moreover, for each $i \in \mathbb{N}, k = 1, \cdots, i$ and $j = 1, \cdots, k$ we have

$$x \in \mathrm{int} J_{i_{k_j}} \Longrightarrow \xi\left(a_{i_k}\right) - \varepsilon < DV_1(x,t) < \xi\left(a_{i_k}\right) + \varepsilon \tag{3.17}$$

Denote the functions $\lambda_1, \mu_1 \in C^0(\Omega \setminus \Gamma_1)$ as

$$\lambda_1(x,t) = \begin{cases} \xi\left(a_{i_k}\right) - 2\varepsilon_{i_k} & if(x,t) \in \mathrm{int}J_{i_{k_j}} \text{ and } J_{i_{k_j}} \cap \mathcal{S} = \emptyset \\ DV_1(x,t) - o_{i_k}(x,t) & if(x,t) \in \mathrm{int}J_{i_{k_j}} \text{ and } J_{i_{k_j}} \cap \mathcal{S} \neq \emptyset \end{cases}$$

and

$$\mu_1(x,t) = \begin{cases} \xi\left(a_{i_k}\right) + 2\varepsilon_{i_k} & if(x,t) \in \mathrm{int}J_{i_{k_j}} \text{ and } J_{i_{k_j}} \cap \mathcal{S} = \emptyset \\ DV_1(x,t) + o_{i_k}(x,t) & if(x,t) \in \mathrm{int}J_{i_{k_j}} \text{ and } J_{i_{k_j}} \cap \mathcal{S} \neq \emptyset \end{cases}$$

Here $o_{i_{k_j}}$ is a real valued continuous function on $\mathbb{R}$ such that $o_{i_{k_j}}(x,0) = 0$ and $0 < o_{i_{k_j}}(x,t) < 2\varepsilon_{i_k}$ $(x,t) \in \Omega$. Therefore it follows from (3.17) that

$$\lambda_1(x,t) < DV_1(x,t) < \mu_1(x,t) \ (x,t) \in \Omega \setminus \Gamma_1$$

and

$$0 \leq \mu_1(x,t) - \lambda_1(x,t) < 4\varepsilon_{i_k} \ (x,t) \in \mathrm{int}I_{i_k}$$

Thus applying (3.13) and proceeding as above we can construct for each $n \in \mathbb{N}$, a sequence of close and nowhere dense set $\Gamma_n \subset \Omega$, a sequence of functions $V_n \in C^1(\Omega \setminus \Gamma_n)$ and functions $\lambda_n, \mu_n \in C^0(\Omega \setminus \Omega_n)$ such that

$$f(x,t) - \frac{\gamma}{n} < TV_n(x,t) < f(x,t) \ (x,t) \in \Omega \setminus \Gamma_n, \tag{3.18}$$

$$\lambda_{n-1}(x,t) \leq \lambda_n(x,t) < DV_n < \mu_n \leq \mu_{n-1} \ (x,t) \in \Omega \setminus \Gamma_n \tag{3.19}$$

and

$$\mu_n - \lambda_n < \frac{4\varepsilon_{i_k}}{n} \ (x,t) \in \left(\mathrm{int}I_{i_k}\right) \cap (\Omega \setminus \Gamma_n).$$

Moreover,

$$D_x V_n(x,0) = \lambda_n(x,0) = \mu_n(x,0) = \frac{d}{dx} u_0(x) \ (x,0) \notin \Gamma \cap \mathcal{S}$$

Set $u_n = (I \circ S)(V_n)$. Then $u_n \in \mathcal{ML}_0^1(\Omega)$. It follows from (3.19) that the functions $\bar{\lambda}_n, \bar{\mu}_n \subset \mathcal{ML}_0^0(\Omega)$ defined as

$$\bar{\lambda}_n = (I \circ S)(\lambda_n), \bar{\mu}_n = (I \circ S)(\mu_n)$$

satisfy

$$\bar{\lambda}_{n-1} \leq \bar{\lambda}_n \leq \mathcal{D}u_n \leq \bar{\mu}_n \leq \bar{\mu}_{n-1}$$

Furthermore, we have

$$\bar{\lambda}_n(x,0) = \mathcal{D}_x u_n(x,0) = \bar{\mu}_n(x,0) = \frac{d}{dx} u_0(x) \ (x,0) \notin \Gamma \cap \mathcal{S} \tag{3.20}$$

and

$$\exists u \in \mathcal{NL}_0 \text{ such that } \sup\{\bar{\lambda}_n : n \in \mathbb{N}\} = u = \inf\{\bar{\mu}_n : n \in \mathbb{N}\}.$$

Therefore the sequence $\{u_n\}$ is a Cauchy sequence in $\mathcal{ML}_0^1(\Omega)$. Moreover, (3.18) implies that $\{Tu_n\}$ converges to $f(x,t)$. It follows from (3.20) that $\{\bar{T}u_n\}$ converges to (f, u_o) in $\mathcal{ML}^0 \times C^1[-a,a]$. This implies that the sequence $\{\bar{T}^\sharp u_n\}$ converges to (f, u_0). By Lemma 2.9 the mapping $\bar{T}^\sharp$ is uniformly continuous. Since u_n is Cauchy, it follows that

$$\begin{aligned} &\exists u^\sharp \in \mathcal{NL}_0^1: \\ &\bar{T}^\sharp u_n \longrightarrow \bar{T}^\sharp u^\sharp = (f, u_0). \end{aligned}$$

This completes the proof.

4 Conclusion

We have shown that the initial value problem (1.1-1.2) admits a generalized solution in the space $\mathcal{NL}_0^1$. Furthermore the the solution satisfies the initial condition in the sense that

$$\begin{array}{ll} \exists & k \in \mathbb{N}: \\ \forall & i = 1 \cdots k: \\ & D_{i,x}^{\#} u^{\#}(x,0) = D_{i,x} u_0(y), y \in (-a,a) \end{array}$$

Moreover, the singularity set

$$\left\{ (y,t) \in \Omega \quad \begin{array}{c} \exists i = 1 \cdots k: \\ D_{i,x}^{\#} u^{\#} \text{ is not continuous at } (y,t) \end{array} \right\}$$

is of first Baire Category in view of property P1 in appendix.

References

[1] Anguelov R. Dedekind order Completion of $C(X)$ by Hausdorff continuous functions. Quaestiones Mathematicae 27 (2004), 152-169.

[2]Anguelov R., Markov S. and Sendov B., The set of Hausdorff continuous functions the largestlinear space of interval functions. Reliable Computing 12 (2006), 337-363.

[3]Anguelov R. and Rosinger E. E. Hausdorff continuous solution of nonlinear PDE through OCM. Quaestiones Mathematicae 28(3) (2005), 271-285.

[4] Arnold V. I. Lectures on PDEs, Springer Universitext, 2004.

[5] Baire R., Lecons sur les fonctions discontinues Collection Borel, Paris (1905).

[6] Barbu V. and Precupanu Th., Convexity and Optimization in Banach Spaces EdituraAcademiei, (1986).

[7] Beattie R. and Butzmann H.P., Convergence Struvtures and Application to Functional Analysis, Klummer Academic Publishers, (2002).

[8] Colombeau J. F. New generalized functions and multiplication of distributions, Noth Holland Mathematics Studies 84, 1984.

[9] Dilworth R. P., The normal completion of the lattice of continuous functions, AMS, Translatios, 68, (1950), 427-438.

[10] Gähler Grundstrukturen der Analysis II, Birkhäuser Verlag, Basel (1978).

[11] Kovalevskaia S. Zur Theorie der partiellen differentialgleichung, Journal für die reine und angewandte Mathematik 80 (1875) 1-32.

[12] Neuberger J. W. Sobolev Gradients and Differential; Equations Springer Lecture Notes in Mathematics, Springer Berlin, 1670, (1997).

[13] Neuberger J. W. Continuous Newton's method for polynimials Math. Intell. 21, (1999), 18 - 23.

[14] Neuberger J. W. A Near Minimal Hypothesis Nash-Moser Theorem, Int. J. Pure Appl. Math., 4, (2003), 269-280.

[15] Neuberger J. W. Prospects of a central theory of partial differential equations, Math. Intell., 27(3), (2005), 47-55.

[16] Oberguggenberger M. B., and Rosinger E. E. Solutions of continuous nonlinear PDEs through order completion North Holland, Amstardan (1994).

[17] Preuß G., Completion of semi-uniform convergence spaces, Appl. Categ. Structures, 8, (2000), 463-473.

[18] Sendov B., Hausdorff Approximations Kluwer Academic, Dordrecht, (1990).

[19] Van der walt, JH. The Order completion method for systems of nonlinear PDEs: Pseudotopological Perspectives. Acta Appl. Math. 103(2008), 1-17.

[20] Van der walt, JH. The uniform order convergence structure on $\mathcal{ML}(X)$ Quaest. Math. 31 (2008), 55-77.

[21] Van der walt, JH. The Order completion method for systems of nonlinear PDEs revisited. Acta Appl. Math. 106 (2009), 149-176.

[22] Van der walt, JH. The completion of Uniform Convergence Spaces and an Application to nonlinear PDEs Quaest. Math. 32 (2009), 371-395.

[23] Wyler Ein Komplettieringsfunktor für uniforme Limesräume, Mathematische Nacherichten 46 (1976), 1-12.