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Generalization of h −Torsion and Curvature Structures in Generalized Recurrent Space of Third Order by Cartan Covariant Derivatives

Adel M. Al-Qashbari [1], Alaa A. Abdallah [*2] & Kamal S. Nasr [3]

[1] *Department of Mathematices, Education Faculty -Aden, University of Aden, Yemen*
Department of Engineering, Faculty of Engineering and Computing,
University of Science & Technology-Aden, Yemen.
[2] *Department of Mathematics, Education Faculty, University of Abyan, Zingibar, Yemen*
[3] *Department of Mathematics, Education Faculty-Adala, University of Aden, Aden, Yemen*
Email: *a.alqashbari@ust.edu,ala733.ala00@gmail.com, kamalsalem1983@gmail.com*

Abstract

In this research, we explore the generalized h −Torsion and curvature structures in Finsler spaces, specifically within the framework of the generalized $U_{|h}$-TRF_n tensor field. By deriving various expressions for these tensors, we reveal the relationships between the generalized h −Torsion tensor and Cartan's curvature tensor. This paper extends the generalized$U_{|h}$-Trirecurrent Finsler space. Further, we obtain some relationships among different curvatures tensors by using Cartan's connection parameter Γ^{*i}_{kh}. The results contribute to the understanding of torsion and curvature in higher-order Finsler spaces and may provide insights into advanced differential geometry and its applications in theoretical physics.

Keywords and phrases:

Generalization generalized h −Trirecurrent space, Curvature tensor U^i_{jkh}, h −TorsionU^i_{jkh}.

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1. Introduction and Preliminaries

Finsler geometry is a generalization of Riemannian geometry, provides a powerful framework for the study of curved spaces where the geometry is defined by a Finsler function. Al-Qashbari et al. [6] have expanded on the concept of generalized Finsler spaces with specific attention to $U\_(|h)$ −birecurrent and recurrent structures. In particular, they introduced an extension of generalized $U\_(|h)$ −birecurrent Finsler space, emphasizing the role of higher-order curvature tensors and their interactions with torsion structures. Furthermore, Al-Qashbariet al.[5] explored the definition of recurrent Finsler structures, incorporating higher-order generalizations through special curvature tensors, which provide a deeper understanding of the curvature relations in these spaces. Similarly, Al-Qashbariet al. [3] have contributed to the study of generalized βK −recurrent Finsler space, focusing on their extensions and developments of the torsion and curvature tensors. Their work highlights the complex relationships between covariant derivatives and the generalized tensors used to describe these spaces. The relationship between two curvature tensors in Finsler spaces have been studied by [1]. Several theorems on curvature tensors obtained by [8 - 13].

Let us consider a set of quantities $g_{ij}(x,y)$defined by [14]

(1.1) $\quad g_{ij}(x\,,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x\,,y),$

whereg_{ij} is positively homogeneous of degree zero in directional arguments.

(1.2) $\quad y_i = g_{ij}\,y^j.$

The quantitiesg_{ij} and g^{jk} are related by

(1.3) $\quad g_{ij}g^{jk} = \delta_i^k = \begin{cases} 1 & if \quad i = k, \\ 0 & if \quad i \neq k. \end{cases}$

In view of (1.1) and (1.3), we get

(1.4) a)$\delta_k^i y_i = y_k$, b) $\delta_k^i y^k = y^i$, c) $\delta_j^i g_{ik} = g_{jk}$, d) $\dot{\partial}_j y^i = \delta_j^i$,

e) $\delta_j^i g^{jk} = g^{ik}$, f) $\delta_j^i \delta_k^j = \delta_k^i$ and g) $\delta_i^i = n$.

The h-covariant derivative of the metric tensor g_{ij} , the associate metric tensorg^{ij} , the vectors y^i and y_i vanish identically, i.e.given by

(1.5) a) $g_{ij|k} = 0$, b) $g^{ij}_{\ \ |k} = 0$, c) $y^i_{\ |k} = 0$ and d) $y_{i\,|k} = 0.$

For an arbitrary vector field X^i, the h-covariant derivative differentiation with respect to x^k, defined above, commutes with the partial differentiation with respect to y^j according to [4, 14]

(1.6) a) $\dot{\partial}_j(X^i_{|k}) - (\dot{\partial}_j X^i)_{|k} = X^r(\dot{\partial}_j \Gamma^{*i}_{rk}) - (\dot{\partial}_r X^i)\, P^r_{jk}$,

whereb) $P^r_{jk} = (\dot{\partial}_j \Gamma^{*r}_{hk}) y^h = \Gamma^{*r}_{jhk}\, y^h$.

The tensor K^i_{jkh} called Cartan's fourth curvature tensor is positively homogeneous of degree zero iny^i and defined by

(1.7)$K^i_{rkh} = \partial_h \Gamma^{*i}_{kr} + (\dot{\partial}_\ell \Gamma^{*i}_{rh}) G^\ell_k + \Gamma^{*i}_{mh}\Gamma^{*m}_{kr} - \partial_k \Gamma^{*i}_{hr} - (\dot{\partial}_\ell \Gamma^{*i}_{rk}) G^\ell_h - \Gamma^{*i}_{mk}\Gamma^{*m}_{hr}$.

The K −Ricci tensor$K_{jk}$and curvature scalar $K$of the curvature tensor K^i_{jkh} are given by

(1.8) $a)$ $K^i_{jkh} = -K^i_{jhk}$, b) $K^i_{jki} = K_{jk}$, c)$g^{jk}K_{jk} = K$

and$d)$ $H^i_{jkh} = K^i_{jkh} + y^s(\dot{\partial}_j K^i_{skh})$.

The tensor R^i_{jkh}called Cartan's third curvature tensor, this tensor is positively homogeneous of degree –1 in $y^i$and skew – symmetric in its last two lower indices k and h and defined by

(1.9) $R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + (\dot{\partial}_\ell \Gamma^{*i}_{jh}) G^\ell_k + G^i_{jm}(\partial_h G^m_k - G^m_{h\ell} G^\ell_k) + \Gamma^{*i}_{mh}\Gamma^{*m}_{jk} - h/k$.

The tensor R^i_{jkh}satisfies the following

(1.10) a) $R^i_{jkh} = -R^i_{jhk}$and b) $R_{ijkh} = g_{rj} R^r_{ikh}$

The tensor H^i_{jkh} is called Berwald curvature tensor. It is positively homogeneous of degree zero in $y^i$and skew-symmetric in its last two lower indices and satisfies the following [2]

(1.11) a) $H^i_{jkh} = -H^i_{jhk}$, b) $H_{jk} = H^i_{jki}$ and c) $H_{jk} y^j = H_k$.

The tensor U^i_{jkh}denoted the tensor $\prod^i_{jkh}$by U^i_{jkh} and defined by

(1.12) $U^i_{jkh} = G^i_{jkh} + \frac{1}{(n+1)}(\delta^i_j G^r_{khr} + y^i G^r_{jkhr})$.

Also, the tensor U^i_{jkh} is called h(v)- Curvature tensor, it is homogeneous of degree -1in$y^i$and symmetric in its last two indices and satisfies the following [15]

(1.13) a) $U^i_{jkh} = U^i_{jhk}$, b) $U^i_{jkh} y^h = U^i_{jhk} y^h = U^i_{jk}$ and c)$U^i_{jhk} y^j = 0$.

The Ricci tensor $U_{jk}$of the projective connection coefficients satisfies the following:

(1.14) a) $U^i_{ikh} = U_{kh}$, b) $U^h_{jkh} = U^h_{jhk} = U_{jk}$ where c) $U_{kh} = \frac{2}{(n+1)} G_{kh}$,

where the tensor $G_{jk}$is Ricci tensor. Thetorsion tensorU^i_{jk} satisfies the following

(1.15) a) $U^i_{jk} = U^i_{kj}$, b) $U^r_{jr} = C^r_{jr}$ and c) $U^r_{jkr} = C^r_{jkr}$.

The generalized $U_{|l}$ −recurrent space, generalized $U_{|l}$ −birecurrent space and generalized $U_{|l}$ −trirecurrent space introduced by [6, 7, 16] which U^i_{jkh} satisfies the following conditions

(1.16) $U^i_{jkh|l} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,$U^i_{jkh} \neq 0$.

(1.17) $U^i_{jkh|l|m} = w_{lm} U^i_{jkh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

(1.18)$U^i_{jkh|l|m|n} = a_{lmn} U^i_{jkh} + b_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Where λ_l and μ_l are non-zero covariant tensors fields of first order,$w_{lm} = \lambda_{l|m} + \lambda_l \lambda_m$ and $v_{lm} = \lambda_l \mu_m + \mu_{l|m}$are non-zero covariant curvature tensors fields of second order. And $a_{lmn} = w_{lm|n} + w_{lm}\lambda_n$ and $b_{lmn} = w_{lm}\, \mu_n + v_{lm|n}$ are non-zero covariant tensors fields of the third order.

Let us consider a Finsler space $F_n$which $h(v)$ −curvature tensor U^i_{jkh} satisfiesa generalization generalized $U_{|h}$ -recurrent Finsler space. i.e. satisfies the following condition [6]

(1.19) $U^i_{jkh|l} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}\eta_l(U^i_k g_{jh} - U^i_h g_{jk})$, $U^i_{jkh} \neq 0$.

where λ_l,μ_l and η_l are non-zero covariant tensors field of first order.

Remark 1.1. If we instead of the division $h(v)$ −curvature tensor U^r_s by other division curvature tensor in (1.19), we get new condition characterized a generalization generalized $U_{|h}$-recurrent Finslerspace.

Taking the h −covariant derivative of first order for (1.19) with respect to x^m using (1.5a), we get

(1.20) $U^i_{jkh|l|m} = w_{lm} U^i_{jkh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4} z_{lm}(U^i_k g_{jh} - U^i_h g_{jk})$

$+\frac{1}{4}\eta_l(U^i_{k|m} g_{jh} - U^i_{h|m} g_{jk})$.

Where $w_{lm} = \lambda_{l|m} + \lambda_l \lambda_m$, $v_{lm} = \lambda_l \mu_m + \mu_{l|m}$ and $z_{lm} = \eta_m \lambda_l + \eta_{l|m}$non-zero covariant tensors of second are order and η_l is non-zero covariant vector of first order.

2. The Extension of Generalized$U_{|h}$– Trirecurrent Finsler Space

In this section we discuss a new development for the generalized $U_{|h}$-Trirecurrent Finslerspace .i.e., we extend the generalized $U_{|h}$-Trirecurrent Finslerspace which characterized by the condition (1.18).

Taking the h −covariant derivative for (1.20) with respect to x^n, we get

(2.1) $U^i_{jkh|l|m|n} = w_{lm|n} U^i_{jkh} + w_{lm}(U^i_{jkh|n}) + v_{lm|n}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$

$+\frac{1}{4} z_{lm|n}(U^i_k g_{jh} - U^i_h g_{jk}) + \frac{1}{4} z_{lm}(U^i_{k|n} g_{jh} - U^i_{h|n} g_{jk})$

$+\frac{1}{4}\eta_{l|n}(U^i_{k|m} g_{jh} - U^i_{h|m} g_{jk}) + \frac{1}{4}\eta_l(U^i_{k|m|n} g_{jh} - U^i_{h|m|n} g_{jk})$.

Using (1.19) in (2.1), we get

(2.2) $U^i_{jkh|l|m|n} = w_{lm|n}U^i_{jkh} + w_{lm}[\lambda_n U^i_{jkh} + \mu_n(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}\eta_n(U^i_k g_{jh} - U^i_h g_{jk})]$

$+ v_{lm|n}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}z_{lm|n}(U^i_k g_{jh} - U^i_h g_{jk}) + \frac{1}{4}z_{lm}(U^i_{k|n} g_{jh} - U^i_{h|n} g_{jk})$

$$+ \frac{1}{4}\eta_{l|n}(U^i_{k|m} g_{jh} - U^i_{h|m} g_{jk}) + \frac{1}{4}\eta_l(U^i_{k|m|n} g_{jh} - U^i_{h|m|n} g_{jk}).$$

Which can be written as

(2.3) $U^i_{jkh|l|m|n} = a_{lmn}U^i_{jkh} + b_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + c_{lmn}(U^i_k g_{jh} - U^i_h g_{jk})$

$+ \frac{1}{4}z_{lm}(U^i_{k|n} g_{jh} - U^i_{h|n} g_{jk}) + \frac{1}{4}e_{ln}(U^i_{k|m} g_{jh} - U^i_{h|m} g_{jk}) + \frac{1}{4}\eta_l(U^i_{k|m|n} g_{jh} - U^i_{h|m|n} g_{jk}).$

Definition 2.1. A Finsler space F_n which $h(v)-$curvature tensor U^i_{jkh} satisfies the condition (2.3), where $a_{lmn} = w_{lm|n} + w_{lm}\lambda_n$, $b_{lmn} = w_{lm}\mu_n + v_{lm|n}$ and $c_{lmn} = \frac{1}{4}(w_{lm}\eta_n + z_{lm|n})$ are non-zero covariant tensors fields of third order, z_{lm} and $e_{ln} = \eta_{l|n}$ is non-zero covariant tensors of second order and η_l is non-zero covariant vector of first order. This space and tensor are called the generalization generalized $U_{|l}$-Trirecurrent space and the tensor will be called the generalization generalized $h-$Trirecurrent tensor, we denote them briefly by $G^{2nd}U_{|l}\text{-}TRF_n$ and $G^{2nd}h\text{-}TR$, respectively.

Transvecting the condition (2.3) by y^h, using(1.2), (1.4b), (1.5c) and (1.13b), we get

(2.4) $U^i_{jk|l|m|n} = a_{lmn}U^i_{jk} + b_{lmn}(\delta^i_k y_j - y^i g_{jk}) + c_{lmn}(U^i_k y_j - U^i g_{jk})$

$+ \frac{1}{4}z_{lm}(U^i_{k|n} y_j - U^i_{|n} g_{jk}) + \frac{1}{4}e_{ln}(U^i_{k|m} y_j - U^i_{|m} g_{jk}) + \frac{1}{4}\eta_l(U^i_{k|m|n} y_j - U^i_{|m|n} g_{jk})$,

Where $U^i_h y^h = U^i$.

Contracting the indices i and k in (2.3) and using (1.4c), (1.4g) and (1.14b), we get

(2.5) $U_{jh|l|m|n} = a_{lmn}U_{jh} + (n-1)b_{lmn}g_{jh} + c_{lmn}(U g_{jh} - U_{hj}) + \frac{1}{4}z_{lm}(U_{|n} g_{jh} - U_{hj|n})$

$+ \frac{1}{4}e_{ln}(U_{|m} g_{jh} - U_{hj|m}) + \frac{1}{4}\eta_l(U_{|m|n} g_{jh} - U_{hj|m|n}),$

where $U^r_r = U$ and $U^r_h g_{jr} = U_{hj}$.

Contracting the indices i and k in (2.4) and using (1.2), (1.4b), (1.4g) and (1.15b), we get

(2.6) $C_{j|l|m|n} = a_{lmn}C_j + (n-1)b_{lmn}y_j + c_{lmn}(U y_j - U_j) + \frac{1}{4}z_{lm}(U_{|n} y_j - U_{j|n})$

$+ \frac{1}{4}e_{ln}(U_{|m} y_j - U_{j|m}) + \frac{1}{4}\eta_l(U_{|m|n} y_j - U_{j|m|n})$,

where $U^r_r = U$, $U^r g_{jr} = U_j$ and $C^r_{jr} = C_j$. Thus, we conclude

Theorem 2.1. *In $G^{2nd}U_{|l}$- TRF_n, the $h(v)$ −torsion tensor U^i_{jk}, U −Ricci tensor U_{jh} and curvature vector C_j are given by (2.4), (2.5) and (2.6), respectively.*

3. Some Relationships Among Curvature Tensors in $G^{2nd}U_{|l}$ −TRF_n

In this section we obtain some relationships among different curvatures tensors by using Cartan's connection parameter Γ^{*i}_{kh}. Let us consider a Finsler space F_n which curvature tensor U^i_{jkh} satisfies the following condition

(3.1) $U^i_{jkh|l} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}\eta_l(K^i_k g_{jh} - K^i_h g_{jk})$, $U^i_{jkh} \neq 0$.

Taking the h −covariant derivative for (3.1) with respect to x^m, using (1.5a), (1.8a) and put $K^i_k g_{jh} = K^i_{jkh}$, we get

(3.2) $U^i_{jkh|l|m} = w_{lm} U^i_{jkh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{2} z_{lm} K^i_{jkh} + \frac{1}{2}\eta_l K^i_{jkh|m}$.

Again, taking the h −covariant derivative for (3.2) with respect to x^n, we get

(3.3) $U^i_{jkh|l|m|n} = w_{lm|n} U^i_{jkh} + w_{lm}(U^i_{jkh|n}) + v_{lm|n}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{2} z_{lm|n}(K^i_{jkh})$

$+\frac{1}{2} z_{lm}(K^i_{jkh|n}) + \frac{1}{2}\eta_{l|n}(K^i_{jkh|m}) + \frac{1}{2}\eta_l(K^i_{jkh|m|n})$.

Suppose that $K^i_k g_{jh} = K^i_{jkh}$ and using (1.8a) in (3.3), we get

(3.4) $U^i_{jkh|l|m|n} - \frac{1}{2}\eta_l K^i_{jkh|m|n} - \frac{1}{2} e_{ln} K^i_{jkh|m} - \frac{1}{2} z_{lm} K^i_{jkh|n}$

$= a_{lmn} U^i_{jkh} + c_{lmn} K^i_{jkh} + b_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Thus, we conclude

Theorem 3.1. *In $G^{2nd}U_{|l}$– TRF_n, the relationship between $h(v)$ −curvature tensor U^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} is given by (3.4).*

Transvecting the condition (3.4) by y^h, using (1.2), (1.4b), (1.5c) and (1.13b), we get

$U^i_{jk|l|m|n} - \frac{1}{2}\eta_l(K^i_{jkh|m|n})y^h - \frac{1}{2} e_{ln}(K^i_{jkh|m})y^h - \frac{1}{2} z_{lm}(K^i_{jkh|n})y^h$

$= a_{lmn} U^i_{jk} + b_{lmn}(\delta^i_k y_j - y^i g_{jk}) + c_{lmn}(K^i_{jkh})y^h$.

Suppose that $K^i_{jkh} y^h = K^i_{jk}$ in the above equation, we get

(3.5) $U^i_{jk|l|m|n} - \frac{1}{2}\eta_l(K^i_{jk|m|n}) - \frac{1}{2} e_{ln}(K^i_{jk|m}) - \frac{1}{2} z_{lm}(K^i_{jk|n})$

$= a_{lmn} U^i_{jk} + b_{lmn}(\delta^i_k y_j - y^i g_{jk}) + c_{lmn}(K^i_{jk})$.

Contracting the indices i and h in (3.4), using (1.4c), (1.4g), (1.8b) and (1.14b) we get

(3.6) $U_{jk|l|m|n} - \frac{1}{2}\eta_l K_{jk|m|n} - \frac{1}{2} e_{ln} K_{jk|m} - \frac{1}{2} z_{lm} K_{jk|n}$

$= a_{lmn} U_{jk} + b_{lmn}(1 - n) g_{jk} + c_{lmn} K_{jk}$.

Thus, we conclude

Corollary 3.1. *The equations (3.5) and (3.6) hold in* $G^{2nd}U_{|l} - TRF_n$.

Let us consider a Finsler space F_n which $h(v)$ −curvature tensor U^i_{jkh} satisfies the following condition

(3.7) $U^i_{jkh|l} = \lambda_l U^i_{jkh} + \mu_l\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\eta_l\left(R^i_k g_{jh} - R^i_h g_{jk}\right),\ U^i_{jkh} \neq 0$.

Taking the h −covariant derivative for (3.7) with respect to x^m and x^n successively, using (1.5a) and (1.10a), we get

(3.8) $U^i_{jkh|l|m|n} = w_{lm|n}U^i_{jkh} + w_{lm}\left(U^i_{jkh|n}\right) + v_{lm|n}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{2}z_{lm|n}\left(R^i_{jkh}\right)$

$+ \frac{1}{2}z_{lm}\left(R^i_{jkh|n}\right) + \frac{1}{2}\eta_{l|n}\left(R^i_{jkh|m}\right) + \frac{1}{2}\eta_l\left(R^i_{jkh|m|n}\right),$

where $R^i_k g_{jh} = R^i_{jkh}$.

Using (3.7) and (1.10a) in the above equation, we get

(3.9) $U^i_{jkh|l|m|n} - \frac{1}{2}\eta_l R^i_{jkh|m|n} - \frac{1}{2}e_{ln}R^i_{jkh|m} - \frac{1}{2}z_{lm}R^i_{jkh|n}$

$= a_{lmn}U^i_{jkh} + c_{lmn}R^i_{jkh} + b_{lmn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right).$

Thus, we conclude

Theorem 3.2. *In* $G^{2nd}U_{|l} - TRF_n$ *,the relationship between* $h(v)$ *−curvature tensor* U^i_{jkh} *and Cartan's third curvature tensor* R^i_{jkh} *is given by(3.9).*

Let us consider a Finsler space F_n which curvature tensor U^i_{jkh} satisfies the following condition

(3.10) $U^i_{jkh|l} = \lambda_l U^i_{jkh} + \mu_l\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\eta_l\left(H^i_k g_{jh} - H^i_h g_{jk}\right),\ U^i_{jkh} \neq 0$.

Taking the h −covariant derivative for (3.10) with respect to x^m and using (1.5a), we get

(3.11) $U^i_{jkh|l|m} = w_{lm}U^i_{jkh} + v_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}z_{lm}\left(H^i_k g_{jh} - H^i_h g_{jk}\right)$

$+ \frac{1}{4}\eta_l\left(H^i_{k|m} g_{jh} - H^i_{h|m} g_{jk}\right)$.

Suppose that $H^i_k g_{jh} = H^i_{jkh}$ and using (1.11a) in (3.11), we get

(3.12) $U^i_{jkh|l|m} = w_{lm}U^i_{jkh} + v_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{2}z_{lm}H^i_{jkh} + \frac{1}{2}\eta_l H^i_{jkh|m}$.

Again, taking the h −covariant derivative for (3.12) with respect to x^n, using (1.5c),(3.11) and (1.11a) in the above equation, we get

(3.13) $U^i_{jkh|l|m|n} = w_{lm|n}U^i_{jkh} + w_{lm}[\lambda_n U^i_{jkh} + \mu_n\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\eta_n\left(H^i_k g_{jh} - H^i_h g_{jk}\right)]$

$+ \frac{1}{2}z_{lm|n}H^i_{jkh} + \frac{1}{2}z_{lm}H^i_{jkh|n} + \frac{1}{2}\eta_l H^i_{jkh|m} + \frac{1}{2}\eta_l H^i_{jkh|m|n}$.

Suppose that $H^i_k g_{jh} = H^i_{jkh}$, we get

(3.14)$U^i_{jkh|l|m|n} - \frac{1}{2}\eta_l H^i_{jkh|m|n} - \frac{1}{2} e_{ln} H^i_{jkh|m} - \frac{1}{2} z_{lm} H^i_{jkh|n}$

$= a_{lmn} U^i_{jkh} + c_{lmn} H^i_{jkh} + b_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Thus, we conclude

Theorem 3.3. *In$G^{2nd}U_{|l} - TRF_n$, the relationship between curvature tensor $U^i_{jkh}$and Berwald's curvature tensor $H^i_{jkh}$is given by (3.14).*

Contracting the indices i and h in (3.14), using (1.4c), (1.4g) and (1.11b), we get

(3.15) $U_{jk|l|m|n} - \frac{1}{2}\eta_l H_{jk|m|n} - \frac{1}{2} e_{ln} H_{jk|m} - \frac{1}{2} z_{lm} H_{jk|n}$

$= a_{lmn} U_{jk} + b_{lmn}(1-n) g_{jk} + c_{lmn} H_{jk}$.

Transvecting the condition (3.15) by y^k, using (1.2), (1.5c) and (1.11c), we get

(3.16) $U_{j|l|m|n} - \frac{1}{2}\eta_l H_{j|m|n} - \frac{1}{2} e_{ln} H_{j|m} - \frac{1}{2} z_{lm} H_{j|n}$

$= a_{lmn} U_j + b_{lmn}(1-n) y_j + c_{lmn} H_j$,

where $U_{jk} y^k = U_j$.Thus, we conclude

Corollary 3.2.*Theequations (3.15) and (3.16)holdin$G^{2nd}U_{|l} - TRF_n$.*

4. Conclusions

The introduction of generalization generalized $U-$Trirecurrent Finsler space has significantly enriched the field of Finsler geometry. We obtained the relationships among different curvatures tensors by using Cartan's covariant derivative in $G^{2nd}U_{|l} - TRF_n$.

Recommendations for Future Research

- Investigate the relationship among different curvature tensors in $G^{2nd}U_{|l} - T\,R\,F_n$.
- Develop numerical methods for studying the properties of generalized U-Trirecurrent Finsler space and generalization generalized$U_{|l}$–Trirecurrent Finsler space.

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