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Berwald Covariant Derivative and Lie Derivative of Conharmonic Curvature Tensors in Generalized Fifth Recurrent Finsler Space

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Abstract

This paper builds upon new define for the conharmonice curvature tensor in generaralized fifth recurrent Finsler space that Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald$(G\mathcal{B}K\text{-}5RF_n)$ via Lie derivative. We define a new conharmonic curvature tensor and explore its relationships with other established curvature tensors. Through various mathematical operations, including the Berwald covariant derivative and the Lie derivative, we derive new expressions for the conharmonic tensor and its interactions with other curvature tensors. The main results include the commutativity of the Berwald covariant derivative of the fifth order with the Lie derivative, as well as the distributivity of the Lie derivative over the addition of curvature tensors. Additionally, we establish several theorems and corollaries that enhance the understanding of the behavior of curvature tensors in $G\mathcal{B}K\text{-}5RF_n$.

Keywords and phrases:

Generalized $\mathcal{B}K$-fifth recurrent Finsler space, Conharmonic curvature tensor L^i_{jkh}, Conformal curvature tensorC_{ijkh}, Lie-derivative L_v .

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1. Introduction and Preliminaries

The study of curvature tensors in Finsler geometry, particularly within the context of recurrent and generalized Finsler spaces, has gained significant attention due to its applications in differential geometry and theoretical physics. In recent years, a number of notable contributions have been made to the understanding of various curvature tensors, their properties, and their interrelations. For instance, (Abdallahal el., 2025) introduced the concept of P −third order generalized Finsler space in the Berwald sense, which laid the foundation for subsequent investigations into higher-order derivatives of curvature tensors. (Ahsan and Ali, 2014) explored properties of the W −curvature tensor, contributing valuable insights into the curvature structure of Finsler spaces.

The concept of conharmonic curvature inheritance has also been studied in the context of spacetime geometry. (Ali et al., 2021) analyzed conharmonic curvature inheritance in the spacetime of general relativity, offering a deeper understanding of the geometric properties in relativistic contexts. On the other hand, (AL-Qashbari et al., 2024) and (AL-Qashbari, 2023) have worked extensively on various aspects of projective curvature tensors and their relationships within recurrent Finsler spaces, shedding light on the intricacies of tensor calculus in higher-dimensional geometries. A significant area of focus has been on the Berwald covariant derivative and Lie-derivatives, which have been explored in multiple studies. Notably, (AL-Qashbari and Baleedi, 2024) have investigated generalized $\mathcal{B}K$-fifth recurrent Finsler spaces ($G\mathcal{B}K$-$5RF_n$), focusing on Lie derivatives of curvature tensors and their interactions. These studies have revealed important identities and relations that influence the geometric behavior of such spaces. Further advancements have been made in identifying the commutativity and distributivity of certain derivatives, thus providing a deeper insight into the structure of these spaces.

(AL-Qashbari and Halboup, 2024), which have explored identities related to Weyl's curvature tensor and conformal curvature tensors, as well as the inheritance of the Kulkarni-Nomizu product in generalized Finsler spaces. The work also draws on insights from other authors such as (Gouin 2023) and (Pak and Kim, 2023), whose studies have extended the Lie derivative's applications in both fluid mechanics and complex geometric structures. Through this research, we seek to offer new perspectives on the differential geometry of higher-order Finsler spaces and to address gaps in the understanding of curvature tensors in these contexts. The present paper builds upon these foundational works, focusing specifically on the conharmonic curvature tensor and its interactions with other curvature tensors in $G\mathcal{B}K\text{-}5RF_n$. Several theorems on curvaturetensors obtained by [1, 2, 3, 6, 11, 17, 19, 21].

Let us explore the infinitesimal transformation point given by

(1.1) $x^{-i} = x^i + v^i(x)\varepsilon$,

whereε is an infinitesimal point constant and $v^i(x)$ is acontravariant vector field independent of directional arguments and dependent on positional coordinatesx^ionly and $v^i(x) \neq 0$. Infinitesimal method is a tool that leads to Lie-derivatives. The Lie-derivative of a vector field x^i in sense of Berwald is given by

(1.2) $L_v x^i = v^j \mathcal{B}_j\, x^i - x^j \mathcal{B}_j v^i + (\dot{\partial}_j\, x^i)\mathcal{B}_s v^j y^s$.

The Lie-derivative of a general mixed tensor field $T^i_{jkh}$is given by

(1.3)$L_v T^i_{jkh} = v^m \mathcal{B}_m T^i_{jkh} - T^m_{jkh}\mathcal{B}_m v^i + T^i_{mkh}\mathcal{B}_j v^m + T^i_{jmh}\mathcal{B}_k v^m$

$+T^i_{jkm}\mathcal{B}_h v^m + \dot{\partial}_m T^i_{jkh}\mathcal{B}_r v^m y^r$.

The Lie-derivative ofthe metric tensors g_{ij} is vanishing

(1. 4)$L_v g_{ij} = 0$.

The Berwaldco variant derivative of the contravariant vector fieldv^m vanish identically, i.e.

(1.5)$\mathcal{B}_j v^m = 0$.

Let us explore a generaralized fifth recurrent Finsler space that Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald($G\mathcal{B}K\text{-}5RF_n$) satisfying the following (AL-Qashbari et al., 2023, 2024)

(1.6) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{sqlnm}\, R^i_{jkh} + b_{sqlnm}\big(\delta^i_h g_{jk} - \delta^i_k\, g_{jh}\big)$

$- c_{sqlnm}\big(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}\big) - d_{sqlnm}\big(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl}\big)$

$- e_{sqlnm}\big(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}\big) - 2b_{qlnm} y^r \mathcal{B}_r\big(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\big)$

if and only if

(1.7)$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m (C^i_{jt} H^t_{kh}) - a_{sqlnm}(C^i_{jt} H^t_{kh}) = 0$.

A conformal curvature tensor C_{ijkh}(also know as Weyl conformal curvature tensor) is defined as

(1.8) $R_{ijkh} = C_{ijkh} + \frac{1}{2}\big(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\big) + \frac{R}{6}\big(g_{ih}g_{jk} - g_{ik}g_{jh}\big)$.

The metric tensor g_{ij} and the Kronecker delta δ^i_h are satisfying the relations:

(1.9) $g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 & , \ if \quad j = k \ , \\ 0 & , \ if \quad j \neq k \ . \end{cases}$

A Finsler space whose Berwald connection parameter G^i_{kh} is independent of y^i is calledanaffinely connected space (Berwald space). Thus, an affinely connected space is characterized by one of the equivalent conditions (Abdallah et al., 2019, 2021, 2022),

(1.10) a)$\mathcal{B}_k g_{ij} = 0$ andb)$\mathcal{B}_k g^{ij} = 0$.

2. Main Results

In this paper, we construct a novel conharmonic curvature tensor tailored for generalized $\mathcal{B}K$-fifth recurrent Finsler spaces. We explore its relationship with other curvature tensors.

A conharmoniccurvature tensor L^i_{jkh} is defined as [5]

(2.1) $L^i_{jkh} = R^i_{jkh} - \frac{1}{2}\left(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k\right)$.

Taking the Lie-derivative of both sides of Eq. (2.1) and using Eqs. (1.3), (1.4), (1.5) and (1.9), where $v^m \neq 0$, we get

$$\mathcal{B}_m L^i_{jkh} = \mathcal{B}_m R^i_{jkh} - \frac{1}{2}\left(g_{jk}\mathcal{B}_m R^i_h - g_{jh}\mathcal{B}_m R^i_k\right) .$$

Taking Berwaldcovariant derivative of fourth order for above equation with respect to x^n, x^l,$x^q$and x^s, using Eq. [(1.10)a], we get

(2.2) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} - \frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{jk}R^i_h - g_{jh}R^i_k\right)$.

Thus, we conclude

Theorem 2.1. *In $G\mathcal{B}K - 5RF_n$, the Berwald covariant derivative of fifth order for the conharmoniccurvature tensor L^i_{jkh} is giving by Eq. (2.2).*

From Eq. (2.2), we get

(2.3) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}$,

if and only if

(2.4) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{jk}R^i_h - g_{jh}R^i_k\right) = 0$.

Therefore, we conclude

Corollary 2.1. *In $G\mathcal{B}K - 5RF_n$, the Berwald covariant derivative of fifth order for the Cartan's third curvature tensor R^i_{jkh} and conharmonic curvature tensor $L^i_{jkh}$are equal if and only if Eq. (2.4) holds.*

Transvecting Eq. (2.1) bythe tensorg^{jm}, using Eq.[(1.9)], we get

(2.5) $L^i_{jkh} = R^i_{jkh}$.

Now, using Eqs. (1.6) and (1.9) in Eq.(2.3), we get

$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\, L^i_{jkh} = a_{sqlnm}R^i_{jkh}$.

Using Eq. (2.5) in above equation, we get

(2.6) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh} = a_{sqlnm}L^i_{jkh}$.

Therefore, we conclude

Theorem 2.2. *In $G\mathcal{B}K - 5RF_n$, the conharmonic curvature tensor L^i_{jkh}behaves as fifth recurrent [provided Eq.(1.7) hold].*

Taking the Lie - derivative of both sides of Eq. (2.2), we get

(2.7) $L_v\big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh}\big) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh})$

$$-\frac{1}{2}L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(g_{jk}R^i_h - g_{jh}R^i_k\big)].$$

Taking the Lie-derivative of both sides of Eq. (2.1), using Eq. (1.9), we get

$$L_v L^i_{jkh} = L_v R^i_{jkh} - \frac{1}{2}L_v\big(g_{jk}R^i_h - g_{jh}R^i_k\big)\,.$$

Taking Berwaldcovariant derivative of fifth order for above equation with respect to x^m, x^n, x^l,$x^q$and x^s, we get

(2.8) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L_v L^i_{jkh}) = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L_v R^i_{jkh}) - \frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big[L_v\big(g_{jk}R^i_h - g_{jh}R^i_k\big)\big]$.

In view of Eqs. (2.7) and (2.8), we get

(2.9) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L_v L^i_{jkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh})$,

if and only if

(2.10) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L_v R^i_{jkh}) - \frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big[L_v\big(g_{jk}R^i_h - g_{jh}R^i_k\big)\big]$

$= L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}) - \frac{1}{2}L_v\big[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(g_{jk}R^i_h - g_{jh}R^i_k\big)\big]$.

Thus, we conclude

Theorem 2.3.*In$G\mathcal{B}K - 5RF_n$, the Lie-derivative for the conharmoniccurvature tensor L^i_{jkh} and Berwald covariant derivative of fifth order are commutative [provided (2.10) holds].*

Using Eqs. (1.6) and (1.9) in Eq. (2.7), we get

$L_v(a_{sqlnm}R^i_{jkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh}) + \frac{1}{2}L_v\big[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(g_{jk}R^i_h - g_{jh}R^i_k\big)\big]$.

Which can be written as

(2.11) $L_v R^i_{jkh} = \frac{1}{(a_{sqlnm})}\big[L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh})$

$+\frac{1}{2}L_v\big[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(g_{jk}R^i_h - g_{jh}R^i_k\big)\big]\big] - \frac{1}{(a_{sqlnm})}R^i_{jkh}L_v\big(a_{sqlnm}\big)$.

Thus, we conclude

Theorem 2.4. *In $G\mathcal{B}K-5RF_n$, the Lie- derivative for the Cartan's third curvature tensor R^i_{jkh} is giving by Eq. (2.11)[provided Eq. (1.7) holds].*

Adding the conharmoniccurvature tensor L^i_{jkh} of both sides of Eq. (1.8), we get

$$L^i_{jkh}+R_{ijkh}=L^i_{jkh}+C_{ijkh}+\frac{1}{2}\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)$$
$$+\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right).$$

Taking Berwaldcovariant derivative of fifth order for above equation, with respect to x^m, x^n, x^l,$x^q$and x^s, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L^i_{jkh}+R_{ijkh}\right)=\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mL^i_{jkh}+\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mC_{ijkh}$$
$$+\frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)$$
$$+\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right].$$

Taking the Lie-derivative of both sides of above equation, we get

(2.12) $$L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L^i_{jkh}+R_{ijkh}\right)\right]=L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mL^i_{jkh}\right)$$
$$+L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mC_{ijkh}\right)+\frac{1}{2}L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)\right]$$
$$+L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right]\right].$$

Which can be written as

(2.13) $$L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L^i_{jkh}+R_{ijkh}\right)\right]=L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mL^i_{jkh}\right)+L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mC_{ijkh}\right),$$

if and only if

(2.14) $$\frac{1}{2}L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih}\right)\right]$$
$$+L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}\left(g_{ih}g_{jk}-g_{ik}g_{jh}\right)\right]\right]=0.$$

Now, transvecting Eq. (1.8) by non-zero tensor ($g^{rk}g^{mh}$) and using Eq. [(1.9)a], we get

(2.15) $C_{ijkh}=R_{ijkh}$.

Using Eq. (2.15) in Eq. (2.13), we get

(2.16) $$L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L^i_{jkh}+C_{ijkh}\right)\right]=L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mL^i_{jkh}\right)+L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mC_{ijkh}\right).$$

Thus, we conclude

Theorem 2.5. *In $G\mathcal{B}K-5RF_n$, the Lie-derivative is distributive on the addition of Berwald covariant derivative of fifth order for the conharmonic curvature tensor L^i_{jkh} and conformal curvature tensor C_{ijkh} [provided Eq.(2.14) holds].*

Using Eq. (2.7) in Eq. (2.12), we get

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L^i_{jkh}+R_{ijkh})]=L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh})$$
$$+L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh})-\frac{1}{2}L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{jk}R^i_h-g_{jh}R^i_k)]$$
$$+\frac{1}{2}L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih})]$$
$$+L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}(g_{ih}g_{jk}-g_{ik}g_{jh})\right]\right],$$

which can be written as

(2.17) $$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L^i_{jkh}+R_{ijkh})]=L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh})+L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh})$$

if and only if

(2.18) $$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{ik}R_{jh}+g_{jh}R_{ik}-g_{ih}R_{jk}-g_{jk}R_{ih})]$$
$$-L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{jk}R^i_h-g_{jh}R^i_k)]+2L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{6}(g_{ih}g_{jk}-g_{ik}g_{jh})\right]\right]=0.$$

Using Eq. (2.5) and Eq. (2.15) in Eq. (2.17), we get

(2.19) $$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^i_{jkh}+R_{ijkh})]=L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh})+L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}).$$

Therefore, we conclude

Corollary 2.2. *In $G\mathcal{B}K-5RF_n$, the Lie- derivative is distributive on the addition of Berwald covariant derivative of fifth order for the Cartan's third curvature tensor R^i_{jkh} and associate curvature tensor R_{ijkh} [provided Eq. (2.18) holds].*

3. Conclusions

In this paper we investigated the commutative relationship between the Lie-derivative andthe Berwald covariant derivative of the fifth order for conharmonic curvature tensor in generalized fifth recurrent Finsler space for K^i_{jkh} in sense of Berwald. We defined conharmonic curvature tensor and established its relations with other tensors via Lie-derivative in the main space. These results highlight significant interrelationships between curvature tensors, derivatives, and the geometry of $G\mathcal{B}K-5RF_n$, providing a deeper understanding of the behavior of these tensors under various operations.

4. Recommendations

We expand this study to include different tensor for various spaces based on the findings derived from the manipulation of the conharmonic curvature tensor L^i_{jkh}, the Cartan's third curvature tensor R^i_{jkh}, and their respective derivatives in the context of $G\mathcal{B}K-5RF_n$. The interrelationships between curvature tensors such as L^i_{jkh}, R^i_{jkh}, and C_{ijkh}suggest promising avenues for understanding the geometric structure of manifolds in the context of $G\mathcal{B}K-5RF_n$ .We can deepen our understanding of the geometric structures in $G\mathcal{B}K-5RF_n$ and potentially unlock new methods for analyzing and modeling complex spaces in theoretical research.

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