e-ISSN 2795-3278
p-ISSN 2795-3274

GPH INTERNATIONAL JOURNAL OF MATHEMATICS
WWW.GPHJOURNAL.ORG

10.5281/zenodo.13384467 Vol. 07 Issue 07 July - 2024 Manuscript ID: #1530

Approximate Lie Symmetries and Conservation Laws of Third-Order Nonlinear Perturbed Korteweg–de Vries Equation

Adnan Shamaoon
Department of Mathematics, Physics and Electrical Engineering, University of Northumbria, Ellison PI, Newcastle upon Tyne NE1 8ST, United Kingdom

Corresponding author: adnan.shamaoon@northumbria.ac.uk

Abstract:

This work analyses the perturbed Korteweg-de Vries (KdV) equation, a third-order nonlinear differential equation that is critical to understanding wave evolution. The emphasis is on discovering and investigating the approximate Lie symmetries and their associated conservation laws with this equation when exposed to different perturbing functions. Using the partial Lagrange approach, the study discovers approximate symmetries and their related conservation laws for the perturbed KdV equation. The goal is to identify particular perturbations that increase the number of approximation symmetries relative to the original KdV equation, exposing previously unknown system properties. The research involves adding various perturbations to the KdV equation, detecting the resulting Lie symmetries, and finding when the perturbed equation exhibits more symmetries than its unperturbed counterpart.

Keywords:

Approximate symmetries, conservation laws, partial differential equations, perturbed KdV equation, wave propagation, and nonlinear dynamics.

1. Introduction

We know that Differential equations (DEs) [1] are essential tools in modelling a vast array of physical phenomena involving changes relative to one or more independent variables, and they are broadly categorised into ordinary differential equations (ODEs) and partial differential equations (PDEs). ODEs involve functions of a single independent variable and their derivatives, making them instrumental in describing systems where changes occur with respect to a single factor, such as time, and are used in simple dynamical systems, population dynamics, and basic physical systems like particle motion under force. PDEs, involving functions of multiple independent variables and their partial derivatives, are more complex and significant in fields such as mathematics, physics, fluid dynamics, mechanics, and physical chemistry, modelling phenomena like heat conduction, wave propagation, fluid flow, and electromagnetic field behaviour. Modelling PDEs under specific conditions and constraints is crucial for effectively manipulating the phenomena they describe; for instance, understanding fluid behaviour under different forces in fluid dynamics or describing wave propagation and heat diffusion in physics. Perturbated partial differential equations (PDEs) [2] are fundamental principles that define essential concepts in the mechanical and chemical engineering disciplines. Third-order derivative perturbed PDEs are particularly useful in mechanical engineering for understanding the dynamics of beam oscillations, stress distribution in complicated materials, and the vibrational behaviour of mechanical systems.

Most real-world problems modelled by PDEs are inherently nonlinear, lacking straightforward analytical solutions, and are tackled using various approximation methods and techniques for high accuracy, such as numerical simulations, perturbation methods, and approximate symmetry techniques [2-4]. Approximate symmetry methods, particularly valuable for nonlinear PDEs, provide a systematic approach to finding approximate solutions by leveraging the symmetries of the equations. Developed by Baikov et al. [5-7] in the 1980s, the method of approximate Lie symmetry extends Lie's theory by incorporating small perturbations, effectively uncovering hidden structures within perturbed PDEs and contributing to the development of conservation laws. The Korteweg–de Vries (KdV) equation [8], a well-known PDE modelling weakly nonlinear long-wavelength waves, describes wave evolution due to the combined effects of weak nonlinearity and dispersion. When perturbed, the KdV equation can be analysed using approximate symmetry methods, allowing researchers to compute approximate symmetries and construct invariant solutions associated with the perturbed equations. These methods offer several advantages, including the identification of additional symmetries not apparent in the exact equation, facilitating the construction of invariant solutions that provide deeper insights into system dynamics, and analyzing system responses to changes through small perturbations to uncover new conservation laws [7, 9-11]. Despite the complexity of computations and the need for careful interpretation of results, the application of approximate symmetry methods remains a powerful tool in the analysis of nonlinear PDEs, offering a pathway to understanding and solving complex dynamical systems more accurately.

Fractional-order partial differential equations (FPDEs) [12, 13] extend traditional PDEs by incorporating non-integer derivatives, enhancing the modelling of complex systems with memory and hereditary properties, such as viscoelastic materials and anomalous diffusion. These equations allow for a more accurate representation of real-world phenomena in fields like mechanics, fluid dynamics, and physical chemistry. For example, the Korteweg–de Vries (KdV) equation, which is a standard PDE used to model wave propagation, can be extended to a fractional KdV equation to describe waves with more complex, real-world characteristics. Similar to perturbation and approximate symmetry methods used for nonlinear PDEs, these techniques also apply to FPDEs, aiding in the discovery of hidden symmetries and conservation laws, thus offering deeper insights and more precise solutions for complex dynamical systems.

This research is designed to gradually investigate both exact and approximate symmetries, as well as tconservation laws derived using them, within the setting of equation (1). Section 2 is devoted to a careful examination of the exact symmetries and exact conservation laws, which provide fundamental insights into the unaltered KdV equation. Moving on, Section 3 digs into the methods designed expressly to address the approximation features of the KdV equation, which are critical when perturbations are applied. This section gives a thorough discussion of the technique, emphasising the changes required to efficiently manage these approximation components. Section 4 applies the previously mentioned technique to several scenarios of the perturbed KdV equation, examining each scenario to obtain the associated conservation laws and proving the method's practical use and relevance. Finally, Section 5 concludes the study by summarising the important findings and emphasising the research's significant insights and contributions to the larger knowledge of nonlinear wave processes.

2. Lie Symmetries of KdV Equation

The symmetries examined in this study [14, 15], are detailed as follows: The focus is on the Korteweg–de Vries (KdV) equation, which is a third-order nonlinear partial differential equation that can be expressed as follows:

$$y_t - 6yy_x + y_{xxx} = 0. \qquad (1)$$

This equation describes the evolution of long, shallow water waves and is a fundamental model in the study of nonlinear wave phenomena [16]. The KdV equation showcases both the effects of nonlinear interactions and dispersion, making it a pivotal subject in mathematical physics. Understanding its exact symmetries and conservation laws is crucial for analyzing the behavior of solutions and developing effective analytical and numerical methods for perturbed systems. The exploration of these symmetries aids in revealing deeper insights into the dynamics represented by the KdV equation.

We define a Lie symmetry generator for equation (1), which takes the following look:

$$\mathcal{X} = \xi^1(x,t,y)\frac{\partial}{\partial x} + \xi^2(x,t,y)\frac{\partial}{\partial t} + \eta(x,t,y)\frac{\partial}{\partial y}. \qquad (2)$$

The highest order of our equation (1) is three, so we extend the symmetry generator in equation (2) to third prolongation.

$$\begin{aligned}\mathcal{X}^{[3]} = \quad & \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y} + \eta_t\frac{\partial}{\partial y_t} + \eta_x\frac{\partial}{\partial y_x} + \eta_{xx}\frac{\partial}{\partial y_{xx}} \\ & + \eta_{xt}\frac{\partial}{\partial y_{xt}} + \eta_{tt}\frac{\partial}{\partial y_{tt}} + \eta_{xxx}\frac{\partial}{\partial y_{xxx}}.\end{aligned} \qquad (3)$$

By applying symmetry generator in equation (3) on equation (1),

$$\mathcal{X}^{[3]}(y_t - 6yy_x + y_{xxxx}) = 0, \quad (4)$$

we obtain

$$\eta_t - 6y\eta_x - 6\eta y_x + \eta_{xxx} = 0. \qquad (5)$$

The expanded form of equation (4) is

$$\begin{aligned} &\left[\eta_t - \xi_t^1 y_x + \left(\eta_y - \xi_t^2\right)y_t - \xi_y^1 y_x y_t - \xi^2 y y_t^2\right] - 6y\left[\eta_x + \left(\eta_y - \xi_x^1\right)y_x - \xi_x^2 y_t - \xi_y^1 y_x^2 - \xi_x^2 y_x y_t\right] \\ &-6\eta y_x + \left[\eta xxx + \left(3\eta_{yxx} - \xi_{xxx}^1\right)y_x - \xi_{xxx}^2 y_t + \left(3\eta_{yyx} - 3\xi_{xxy}^1\right)\right. \\ &y_x^2 - 3\xi_{yxx}^2 y_x y_t + \left(3\eta_{yx} - 3\xi_{xx}^1\right)y_{xx} - 3\xi_{xx}^2 y_{xt} + \left(\eta_{yyy} - \xi_{yyx}^1\right)y_x^3 \\ &-3\xi_{yyx}^2 y_x^2 y_t + \left(3\eta_{yy} - 9\xi_{yx}^1\right)y_x y_{xx} - 6\xi_{xy}^2 y_x y_{xt} - 3\xi_{yx}^2 y_t y_{xx} + \left(\eta_y - 3\xi_x^1\right)y_{xxx} \\ &-3\xi_x^2 y_{xxt} - \xi_{yyy}^1 y_x^4 - \xi_{yyy}^2 y_x^3 y_t - 6\xi_{yy}^1 y_x^2 y_{xxx} - 3\xi_{yy}^2 y_x^2 y_{xt} - 3\xi_{yy}^2 y_x y_t y_{xxx} \\ &-4\eta_y y_x y_{xxxx} - 3\xi_y^2 y_x y_{xxt} - 3\xi_y^1 y_{xx}^2 - 3\xi_y^2 y_{xx} y_{xt} - \xi_y^2 y_t y_{xxxx} = 0. \end{aligned} \qquad (6)$$

When we substitute equation (1) in equation (6), we get the following equations:

$$\begin{aligned} &\eta_t - \xi_t^1 y_x + \left(\eta_y - \xi_t^2\right)y_t - \xi_y^1 y_x y_t - \xi_y^2 y_t^2 - 6\eta y_x - 6y\left[\eta_x + \left(\eta_y - \xi_x^1\right)y_x - \xi_x^2 y_t - \xi_y^1 y_x^2 - \xi_y^2 y_x y_t\right] \\ &+\eta_{xxx} + \left(3\eta_{xxy} - \xi_{xxx}^1\right)y_x - \xi_{xxx}^2 y_t + \left(3\eta_{xyy} - 3\xi_{xxy}^1\right)y_x^2 \\ &-3\xi_{xxy}^2 y_x y_t + \left(3\eta_{xy} - 3\xi_{xx}^1\right)y_{xx} - 3\xi_{xx}^2 y_{xt} + \left(\eta_{yyy} - \xi_{xyy}^1\right)y_x^3 - 3\xi_{xyy}^2 y_x^2 y_t \\ &+\left(3\eta_{yy} - 9\xi_{xy}^1\right)y_x y_{xx} - 6\xi_{xy}^2 y_x y_{xt} - 3\xi_{xy}^2 y_t y_{xx} + 6y\left(\eta_y - 3\xi_x^1\right)y_x - \left(\eta_y - 3\xi_x^1\right)y_t \\ &-3\xi_x^2 y_{xxt} - \xi_{yyy}^1 y_x^4 - \xi_{yyy}^2 y_x^3 y_t - 6\xi_{yy}^1 y_x^2 y_{xx} - 3\xi_{yy}^2 y_x^2 y_{xt} - 3\xi_{yy}^2 y_x y_t y_{xxx} \\ &-24y\xi_y^1 y_x^2 + 4\xi_y^1 y_x y_t - 3\xi_y^2 y_x y_{xxt} - 3\xi_y^1 y_{xx}^2 - 3\xi_y^2 y_{xx} y_{xt} - 6y\xi_y^2 y_x y_{xt} + \xi_y^2 y_t^2 = 0. \end{aligned} \qquad (7)$$

All derivative terms of y are independent in equation (7). When we compare the coefficients of derivative terms of y, we get the following equations and monomials, as shown in Table 1.

Table 1: The exact symmetries of equation (1).

Coefficients	Monomials
$\eta_t - 6y\eta_x + \eta_{xxx} = 0$	1
$-\xi_t^1 - 6\eta - 6y\eta_y - \xi_x^1 + 3\eta_{xxy} - \xi_{xxx}^1 + 6y\eta_y - 3\xi_x^1 = 0$	y_x
$-\xi_y^1 + 6y\xi_y^2 - 3\xi_{xxy}^2 + 4\xi_y^1 = 0$	$y_x y_t$
$\eta_y - \xi_t^2 + 6y\xi_x^2 - \xi_{xxx}^2 - \eta_y - 3\xi_x^1 = 0$	y_t
$-\xi_y^2 + \xi_y^2 = 0$	y_t^2
$6y\xi_y^1 + 3\eta_{xyy} - 3\xi_{xxy}^1 - 24y\xi_y^1 = 0$	y_x^2
$3\eta_{xy} - 3\xi_{xx}^1 = 0$	y_{xx}
$-3\xi_{xx}^2 = 0$	y_{xt}
$\eta_{yyy} - \xi_{xyy}^1 = 0$	y_x^3
$-3\xi_{xyy}^2 = 0$	$y_x^2 y_t$
$3\eta_{yy} - 9\xi_{xy}^1 = 0$	$y_x y_{xx}$
$-6\xi_{xy}^2 - 6y\xi_y^2 = 0$	$y_x y_{xt}$
$-3\xi_{xy}^2 = 0$	$y_t y_{xx}$
$-3\xi_x^2 = 0$	y_{xxt}
$\xi_{yyy}^1 = 0$	y_x^4
$\xi_{yyy}^2 = 0$	$y_x^3 y_t$
$\xi_{yy}^1 = 0$	$y_x^2 y_{xx}$
$\xi_{yy}^2 = 0$	$y_x^2 y_{xt}$

$\xi^2_{yy} = 0$	$y_x y_t y_{xx}$
$\xi^2_y = 0$	$y_x y_{xxt}$
$\xi^1_y = 0$	y^2_{xx}
$\xi^2_y = 0$	$y_{xx} y_{xt}$

From the Table 1 we get the set of PDEs that are required which are listed as follows:

$$\xi^2_y = 0, \tag{8}$$

$$\xi^1_y = 0, \tag{9}$$

$$\xi^2_x = 0, \tag{10}$$

$$3\xi^1_x - \xi^2_t = 0. \tag{11}$$

Form equation (11),

$$\begin{aligned} \xi^1_{xx} &= 0, \\ \Rightarrow \eta_{xxy} &= \xi^1_{xxx}, \\ \Rightarrow \eta_{xxy} &= 0, \\ \eta &= \frac{-1}{6}\xi^1_t - 2y\xi^1_x. \end{aligned} \tag{12}$$

As we know

$$\xi^1_x = \frac{1}{3}\xi^2_t, \tag{13}$$

hence,

$$\begin{aligned} \eta &= -\frac{1}{6}\xi^1_t - \frac{2}{3}y\xi^2_t, \\ \eta_t &= -\frac{1}{6}\xi^1_{tt} - \frac{2}{3}y\xi^2_{tt}, \\ \eta_x &= -\frac{1}{6}\xi^1_{xt} - \frac{2}{3}y\xi^2_{tx}, \\ \eta_{xxx} &= 0, \end{aligned} \tag{14}$$

$$\xi^1_{tt} = 0, \tag{15}$$

$$\xi^2_{tt} = 0. \tag{16}$$

Suppose

$$\begin{aligned} \xi^2 &= F(t), \\ \Rightarrow F_{tt}(t) &= 0. \end{aligned} \tag{17}$$

When we solve above differential equation in equation (17), we get

$$\Rightarrow F_t(t) = c_1, \quad \Rightarrow F(t) = c_1 t + c_2. \tag{18}$$

We get first general solution, which is:

$$\xi^2 = c_1 t + c_2. \tag{19}$$

From equation (11),

$$3\xi_x^1 - \xi_t^2 = 0, \quad \xi_y^1 = 0 \Rightarrow \xi^1 = G(x)t, \quad \xi_x^1 = \frac{1}{3}\xi_t^2, \tag{20}$$

substitute the value of ξ^2 from equation (18)

$$\xi_x^1 = \frac{1}{3}c_1. \tag{21}$$

When we solve equation (20) by taking integration w.r.t. x, we get

$$\Rightarrow \xi^1 = \frac{1}{3}c_1 x + H(t), \quad \Rightarrow \xi_{tt}^1 = H_{tt}(t) = 0, \quad \Rightarrow H(t) = c_3 t + c_4. \tag{22}$$

Finally, we obtain our next solution, which is:

$$\xi^1 = \frac{1}{3}c_1 x + c_3 t + c_4. \tag{23}$$

We use first equation from equation (14), which gives us the following solution:

$$\eta = -\frac{1}{6}c_3 - \frac{2}{3}c_1 y. \tag{24}$$

The Lie symmetry generators in (2) gives the following Lie symmetries for equation (1) are by using constants $c_i, i = 1,2,3,4$, given in Table 2.

Table 2: Lie symmetries of equation (1).

Lie symmetries
$\mathcal{X}_1 = \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}y\frac{\partial}{\partial y}$
$\mathcal{X}_2 = \frac{\partial}{\partial t}$
$\mathcal{X}_3 = t\frac{\partial}{\partial x} - \frac{1}{6}\frac{\partial}{\partial y}$
$\mathcal{X}_4 = \frac{\partial}{\partial x}$

3. Approximate Lie Symmetries of Perturbed KdV equation

This section outlines the development of a different way to find the approximate symmetries [2] of equation (1) that is perturbed with the function $p(x,t,y(x,t),y(t,x))$ and defined as:

$$y_t - 6yy_x + y_{xxx} + \varepsilon p(x,t,y(x,t) = 0, \qquad (25)$$

where ε is a small parameter that introduces the necessary perturbation to the KdV equation. We write separately both exact and approximate components of equation (25) are

$$\begin{aligned} \mathcal{E}_e &= y_t - 6yy_x + y_{xxx}, \\ \mathcal{E}_p &= p(x,t,y(x,t)). \end{aligned} \qquad (26)$$

Equation (25) can be written in the following compact form

$$\mathcal{E}_e + \varepsilon\mathcal{E}_p = 0. \qquad (27)$$

On the same way, we write the exact and approximate Lie symmetries as in the following new form

$$\mathcal{X} = \mathcal{X}_e + \varepsilon\mathcal{X}_p. \qquad (28)$$

Here,

$$\mathcal{X}_e = \xi_e^1 \frac{\partial}{\partial x} + \xi_e^2 \frac{\partial}{\partial t} + \eta_e \frac{\partial}{\partial y}, \qquad (29)$$

Represents the exact part of Lie symmetry generator, and

$$\mathcal{X}_p = \xi_p^1 \frac{\partial}{\partial x} + \xi_p^2 \frac{\partial}{\partial t} + \eta_p \frac{\partial}{\partial y}, \qquad (30)$$

is the approximate part of the Lie symmetry generator. Moreover, ξ^1, ξ^2, and η are the unknown functions which are depending on x, t, and y, respectively. We use the generator $\mathcal{X}$ defined in equation (28) on equation (27), we have

$$(\mathcal{X}_e + \varepsilon\mathcal{X}_p)(\mathcal{E}_e + \varepsilon\mathcal{E}_p) = 0, \qquad (31)$$

that gives

$$\mathcal{X}_e\mathcal{E}_e + \varepsilon(\mathcal{X}_p\mathcal{E}_e + \mathcal{X}_e\mathcal{E}_p) + O(\varepsilon^2) = 0. \qquad (32)$$

The comparison of coefficients of ε^0 and ε^1, respectively, yields the exact and approximate symmetries of the corresponding PDEs as in the following:

$$\begin{aligned} \mathcal{X}_e\mathcal{E}_e &= 0, \\ \mathcal{X}_p\mathcal{E}_e + \mathcal{X}_e\mathcal{E}_p &= 0. \end{aligned} \qquad (33)$$

This equation also shows the approximate Lie symmetries, which will be used to find thefunction $p(x,t,y(x,t))$ [14, 17] as well as assist in deriving the approximate conservation laws associated with the KdV equation's dynamics.

4. New method to find approximate Lie Symmetries and Conservation Laws

We use a newly developed method to get the approximate symmetries and this method discusses the different cases to find the associated conservation laws of the following perturbed KdV equation [9, 18, 19],

$$y_t - 6yy_x + y_{xxx} + \varepsilon p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x) = 0. \quad (34)$$

By utilizing the method outlined in [20-22] for expanding y, and we have

$$y = \chi + \varepsilon\tau. \quad (35)$$

Using this expansion in equation (34),

$$\begin{aligned} (\chi_t + \varepsilon\tau_t) - 6(\chi + \varepsilon\tau)(\chi_x + \varepsilon\tau_x) + (\chi_{xxxx} + \varepsilon\tau_{xxx}) &= \varepsilon p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x), \\ \chi_t + \varepsilon\tau_t - 6\chi\chi_x - 6\varepsilon\chi\tau_x - 6\varepsilon\tau\chi_x - 6\varepsilon^2\tau\tau_x + \chi_{xxx} + \varepsilon\tau_{xxx} &= \varepsilon p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x), \\ (\chi_t - 6\chi\chi_x + \chi_{xxx}) + \varepsilon(\tau_t - 6\chi\tau_x - 6\tau\chi_x + \tau_{xxx}) + \varepsilon^2(-6\tau\tau_x) &= \varepsilon p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x). \end{aligned} \quad (36)$$

Equation (36) can be written in the following form by neglecting higher order terms of ε

$$\Lambda_e + \varepsilon\Lambda_p = 0. \quad (37)$$

When we compare the coefficients of ε^0 and ε^1 in equation (36), we get the following system

$$\begin{aligned} \Lambda_e &:= \chi_t - 6\chi\chi_x + \chi_{xxx} = 0, \\ \Lambda_p &:= \tau_t - 6\chi\tau_x - 6\tau\chi_x + \tau_{xxx} - p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x) = 0. \end{aligned} \quad (38)$$

The approximate Lie symmetry operator/generator is written as

$$\mathcal{X} = \mathcal{X}_e + \varepsilon\mathcal{X}_p = 0. \quad (39)$$

Here,

$$\begin{aligned} \mathcal{X}_e &= \xi_e^1\frac{\partial}{\partial x} + \xi_e^2\frac{\partial}{\partial t} + \eta_e\frac{\partial}{\partial \chi} + \xi_e^1\frac{\partial}{\partial \tau}, \\ \mathcal{X}_p &= \xi_p^1\frac{\partial}{\partial x} + \xi_p^2\frac{\partial}{\partial t} + \eta_p\frac{\partial}{\partial \chi} + \xi_p^1\frac{\partial}{\partial \tau}. \end{aligned} \quad (40)$$

Applying the Lie generator,

$$\begin{aligned} \mathcal{X}(\Lambda_e + \varepsilon\Lambda_p) &= 0, \\ (\mathcal{X}_e + \varepsilon\mathcal{X}_p)(\Lambda_e + \varepsilon\Lambda_p) &= 0. \end{aligned} \quad (41)$$

which gives us

$$\begin{aligned} \mathcal{X}_e\Lambda_e + \varepsilon(\mathcal{X}_p\Lambda_e + \mathcal{X}_e\Lambda_p) + o(\varepsilon^2) &= 0, \\ \mathcal{X}_e\Lambda_e &= 0, \\ \mathcal{X}_p\Lambda_e + \mathcal{X}_e\Lambda_p &= 0. \end{aligned} \quad (42)$$

Let us now go over the following scenarios in further depth.

Case I.

Let, we define

$$p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x) = -\chi_t - \tau_t. \qquad (43)$$

We get the following system of determining PDEs from equation (38),

$$\begin{aligned} &\xi^2_{tt} = 0, \quad \xi^2_{\chi} = 0, \quad \xi^1_t = 0, \\ &\xi^2_{\tau} = 0, \quad \xi^1_{\tau} = 0,, \quad \xi^1_{\chi} = 0, \\ &\xi^2_x = 0, \quad \xi^1_x = \frac{3}{\xi^2_t}, \ \eta = \frac{-2}{3}\chi\xi^2_t, \\ &\xi^1 = -\frac{2}{3}\xi^2_t\tau. \end{aligned} \qquad (44)$$

As

$$\begin{aligned} &\frac{\partial\xi^2}{\partial\chi} = 0, \\ &\frac{\partial\xi^2}{\partial\tau} = 0, \\ &\frac{\partial\xi^2}{\partial x} = 0, \end{aligned} \qquad (45)$$

this suggests that ξ^2 depends solely on t. As a result,

$$\xi^2_{tt} = 0. \qquad (46)$$

By integrating this equation twice with respect to t, we obtain

$$\xi^2 = c_1 t + c_2, \qquad (47)$$

and also,

$$\begin{aligned} &\frac{\partial\xi^1}{\partial t} = 0, \\ &\frac{\partial\xi^1}{\partial\chi} = 0, \\ &\frac{\partial\xi^1}{\partial\tau} = 0, \end{aligned} \qquad (48)$$

which shows that ξ^1 is the function of x alone. Therefore,

$$\frac{\partial\xi^1}{\partial x} = \frac{1}{3}\xi^2 t. \qquad (49)$$

Putting the value of ξ^2_t in equation (49), we get

$$\frac{\partial\xi^1}{\partial x} = \frac{1}{3}c_1. \qquad (50)$$

Integrating equation (50), we get

$$\xi^1 = \frac{1}{3}c_1 x + c_3. \qquad (51)$$

Now,

$$\eta = -\frac{2}{3}\chi\xi_t^2. \quad (52)$$

Putting the value of " ξ_t^2 " in equation (52),

$$\eta = -\frac{2}{3}\chi c_1. \quad (53)$$

By taking

$$\xi^1 = -\frac{2}{3}\xi_t^2\tau, \quad (54)$$

and putting the value of " ξ_t^2 " in equation (54),

$$\xi^1 = -\frac{2}{3}c_1\tau. \quad (55)$$

Therefore,

$$\begin{gathered} \xi^1 = \frac{1}{3}c_1 x + c_3, \quad \xi^2 = c_1 t + c_2, \\ \eta = \frac{-2}{3}\chi c_1, \quad \xi^1 = \frac{-2}{3}c_1\tau. \end{gathered} \quad (56)$$

When we use constants $c_i, i = 1,2,3$, we get the approximate Lie symmetries, which aregiven in Table 3.

Table 3: Approximate Lie symmetries of equation (34).

Approximate Lie symmetries
$\mathcal{X}_1 = \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}\frac{\partial}{\partial \tau} - \frac{2}{3}\chi\frac{\partial}{\partial \chi}$
$\mathcal{X}_2 = \frac{\partial}{\partial t}$
$\mathcal{X}_3 = \frac{\partial}{\partial x}$

4.1. Conservation Laws of Perturbed KdV equation

We find the conservation laws of equation (34) in the following ways:

$$\begin{gathered} \mathcal{X}_1(\psi(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x)) = 0, \\ \left(\frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}\frac{\partial}{\partial \chi} - \frac{2}{3}\chi\frac{\partial}{\partial \chi}\right)\psi = 0, \\ \frac{1}{3}x\psi_x + t\psi_t - \frac{2}{3}\psi_\tau - \frac{2}{3}\chi\psi_\chi = 0, \\ 3\frac{dx}{x} = \frac{dt}{t} = \frac{d\tau}{(-2/3)} = \frac{-3}{2}\frac{d\chi}{\chi} = \frac{d\psi}{0}. \end{gathered} \quad (57)$$

Now, by taking

$$\eta = \frac{-2}{3}c_1\chi - \frac{1}{6}c_4,$$
$$\xi^1 = 6c_3\tau + c_3,$$
$$\xi^2 = c_1 t + c_2, \qquad (64)$$
$$\xi^1 = \frac{1}{3}c_1 x + c_4 t + c_5.$$

Using the $c_i, i = 1, \ldots, 5$, we get approximate Lie symmetries and their associated conservation laws that are given in Table 4.

Table 4: Approximate Lie symmetries and associated conservation laws.

Approximate Lie symmetries	Associated conservation laws
$X_1 = \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}\chi\frac{\partial}{\partial \chi}$	$\psi_1 = \frac{x^3}{t} + x\sqrt{\chi}^{-3} + t\chi^{\frac{3}{2}}$
$X_2 = \frac{\partial}{\partial t}$	$\psi_2 = fx, y, \tau, \tau_t, \tau_x, \chi_t, \chi_x$
$X_3 = 6\tau + 1\frac{\partial}{\partial \tau}$	$\psi_3 = gx, y, t, \tau_t, \tau_x, \chi_t, \chi_x$
$X_4 = t\frac{\partial}{\partial x} - \frac{1}{6}\frac{\partial}{\partial \chi}$	$\psi_4 = \frac{x}{t} + 6\chi$
$X_5 = \frac{\partial}{\partial x}$	$\psi_5 = hy, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x$

Case 3.

In this case, we define

$$p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x) = -\tau_x. \qquad (65)$$

From equation (38), we get after comparing the coefficients of ε^0 and ε^1,

$$\chi_t - 6\chi\chi_x + \chi_{xxx} = 0,$$
$$\tau_t - 6\chi\tau_x - 6\tau\chi_x + \tau_{xxx} - \tau_x = 0. \qquad (66)$$

We find the following system of determining equations:

$$\xi_t^1 = 0,\ \xi_{tt}^1 = 0,\ \xi_\chi^1 = 0,$$
$$\xi_x^1 = 0,\ \xi_\tau^1 = \frac{\xi^1}{\tau},\ \xi_\tau^1 = 0,$$
$$\xi_x^1 = 0,\ \xi_t^2 = 0,\ \xi_\chi^1 = 0, \qquad (67)$$
$$\xi_\chi^2 = 0,\ \eta = \frac{-1}{6}\xi_t^1\xi_\tau^2 = 0,\ \xi_x^2 = 0.$$

We get following solution

$$\eta = \frac{-1}{6}c_1,$$
$$\xi^1 = c_3\tau,$$
$$\xi^2 = c_4, \quad (68)$$
$$\xi^1 = c_1 t + c_2.$$

The approximate Lie symmetries and their associated conservation laws for Case 3 are presented below.

Case 4

In this case, we take

$$p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x) = \chi\tau, \quad (69)$$

the system of equations (38) will take the following form

$$\chi_t - 6\chi\chi_x + \chi_{xxx} = 0,$$
$$\tau_t - 6\chi\tau_x - 6\tau\chi_x + \tau_{xxx} + \chi\tau = 0. \quad (70)$$

Applying equation (39) to equation (70), we get the following set of PDEs:

$$\xi_t^2 = 0,\ \xi_t^1 = \frac{1}{6}\tau\xi_t^1,\ \xi_x^1 = 0,$$
$$\xi_\chi^1 = 0,\ \xi_x^2 = 0,\ \xi_\tau^1 = \frac{\xi^1}{\tau},$$
$$\xi_{tt}^1 = 0,\ \xi_x^1 = 0,\ \xi_\chi^2 = 0, \quad (71)$$
$$\xi_\chi^1 = 0,\ \xi_\tau^2 = 0,\ \xi_\tau^1 = 0,$$
$$\eta = -\frac{1}{6}\xi_t^1.$$

The above equations yield

$$\xi^1 = c_1 t + c_2,$$
$$\xi^2 = c_3,$$
$$\eta = \frac{-1}{6}c_1, \quad (72)$$
$$\xi^1 = \frac{1}{6}\tau c_1 t + c_4.$$

We get the approximate Lie symmetries and their associated conservation laws of Case 4 by applying conditions on constants c_i.

Case 5

Let, we define

$$p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x) = -\tau, \quad (73)$$

and we get the following the system

$$\chi_t - 6\chi\chi_x + \chi_{xxx} = 0,$$
$$\tau_t - 6\chi\tau_x - 6\tau\chi_x + \tau_{xxx} - \tau = 0. \quad (74)$$

Applying equation (39) to equation (74) gives the following set determining of PDEs:

$$\begin{aligned}&\xi_t^1 = \xi_t^2\tau,\ \xi_\tau^1 = 0,\ \xi_\chi^1 = 0,\\ &\xi_\chi^1 = 0,\ \xi_\tau^1 = \frac{\xi^1}{\tau},\ \xi_{tt}^2 = 0,\\ &\xi_x^1 = 0,\ \xi_\chi^2 = 0,\ \xi_x^1 = \frac{1}{3}\xi_t^2\\ &\xi_\tau^2 = 0,\ \xi_x^2 = 0.\end{aligned} \tag{75}$$

Solving the above equations, we get

$$\begin{aligned}&\xi^1 = \frac{c_1 x}{3} + tc_3 + c_4,\\ &\xi^2 = c_1 t + c_2,\\ &\eta = \frac{-2}{3}\chi\xi_t^2 c_1 - \frac{1}{6}\xi_t^1 c_3,\\ &\xi^1 = \tau(c_1 t + c_5).\end{aligned} \tag{76}$$

Finally, we find approximate symmetries and associated conservation laws of Case 5 when we apply conditions on c_i.

Case 6

We assume that

$$p(x, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x) = -\tau\chi_t, \tag{77}$$

then the system of equations (38) take the following form

$$\begin{aligned}&\chi_t - 6\chi\chi_x + \chi_{xxx} = 0,\\ &\tau_t - 6\chi\tau_x - 6\tau\chi_x + \tau_{xxx} - \tau\chi_t = 0.\end{aligned} \tag{78}$$

Applying equation (39) to equation (78) results in the following set of PDEs:

$$\begin{aligned}&\xi_{tx}^2 = 0,\ \xi_t^1 = 0,\ \xi_\chi^1 = 0,\\ &\xi_\tau^1 = 0,\ \xi_\tau^1 = \frac{\xi^1}{\tau},\ \xi_\chi^1 = 0,\\ &\xi_\tau^2 = 0,\ \xi_x^1 = 0,\ \xi_t^2 = 0,\\ &\xi_t^1 = 0,\ \xi_\chi^2 = 0.\end{aligned} \tag{79}$$

When we solve the above set of equations, we find the following solutions

$$\begin{aligned}&\xi^1 = c_3,\\ &\xi^2 = c_2,\\ &\eta = 0,\\ &\xi^1 = c_1\tau.\end{aligned} \tag{80}$$

As a result, we get approximate Lie symmetries and their conservation laws of Case 6 with the help of conditions on c_i.

Table 5: Approximate Lie symmetries and associated conservation laws.

Approximate Lie symmetries	Associated conservation laws
$\mathcal{X}_1 = \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \tau t\frac{\partial}{\partial \tau} - \frac{2}{3}\chi\frac{\partial}{\partial \chi}$	$\psi_1 = \frac{x^3}{t} + x^3 + x^3\chi^{\frac{3}{2}} + e^t\tau + t\chi^{\frac{3}{2}}$
$\mathcal{X}_2 = \frac{\partial}{\partial t}$	$\psi_2 = fx, y, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x$
$\mathcal{X}_3 = \frac{\partial}{\partial x} - \frac{1}{6}\frac{\partial}{\partial \chi}$	$\psi_3 = x + 6\chi$
$\mathcal{X}_4 = \frac{\partial}{\partial x}$	$\psi_4 = gy, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x$
$\mathcal{X}_5 = \tau\frac{\partial}{\partial \tau}$	$\psi_5 = hx, y, t, \tau_t, \tau_x, \chi_t, \chi_x$

Table 6: Approximate Lie symmetries and associated conservation laws.

Approximate Lie symmetries	Associated conservation laws
$\mathcal{X}_1 = \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}\frac{\partial}{\partial \tau} - \frac{2}{3}\chi\frac{\partial}{\partial \chi}$	$\psi_1 = \frac{x^3}{t} + x^3 e^{\frac{3}{2}}\tau + x^3\chi^{\frac{3}{2}} + te^{\frac{3}{2}}\tau + t\chi^{\frac{3}{2}} + \chi e^{-\tau}$
$\mathcal{X}_2 = \frac{\partial}{\partial t}$	$\psi_2 = fx, y, \tau, \tau_t, \tau_x, \chi_t, \chi_x$
$\mathcal{X}_3 = \frac{\partial}{\partial x}$	$\psi_3 = gy, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x$

Table 7: Approximate Lie symmetries and associated conservation laws.

Approximate Lie symmetries	Associated conservation laws
$\mathcal{X}_1 = -\frac{1}{6}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$	$\psi_1 = e^{-6x}/t$
$\mathcal{X}_2 = x\frac{\partial}{\partial x}$	$\psi_2 = fy, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x$
$\mathcal{X}_3 = \tau\frac{\partial}{\partial \tau}$	$\psi_3 = gx, y, t, \tau_t, \tau_x, \chi_t, \chi_x$
$\mathcal{X}_4 = \frac{\partial}{\partial t}$	$\psi_4 = hx, y, \tau, \tau_t, \tau_x, \chi_t, \chi_x$

Table 8: Approximate Lie symmetries and associated conservation laws.

Approximate Lie symmetries	Associated conservation laws
$\mathcal{X}_1 = t\frac{\partial}{\partial x} + \frac{1}{6}\tau t\frac{\partial}{\partial \tau} - \frac{1}{6}\frac{\partial}{\partial \chi}$	$\psi_1 = e^{\frac{x}{t}}/\tau^6 + \tau e^{\chi} + \frac{x}{t} + 6\chi$
$\mathcal{X}_2 = \frac{\partial}{\partial x}$	$\psi_2 = fy, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x$
$\mathcal{X}_3 = \frac{\partial}{\partial t}$	$\psi_3 = gx, y, \tau, \tau_t, \tau_x, \chi_t, \chi_x$
$\mathcal{X}_4 = \tau\frac{\partial}{\partial \tau}$	$\psi_4 = hx, y, t, \tau_t, \tau_x, \chi_t, \chi_x$

Table 9: Approximate Lie symmetries and associated conservation laws

Approximate Lie symmetries	Associated conservation laws
$\mathcal{X}_1 = \tau\frac{\partial}{\partial \tau}$	$\psi_1 = fx, y, t, \tau_t, \tau_x, \chi_t, \chi_x$
$\mathcal{X}_2 = \frac{\partial}{\partial t}$	$\psi_2 = gx, y, \tau, \tau_t, \tau_x, \chi_t, \chi_x$
$\mathcal{X}_3 = \frac{\partial}{\partial x}$	$\psi_3 = hy, \tau, t, \tau_t, \tau_x, \chi_t, \chi_x$

5. Conclusion

The third-order nonlinear KdV equation (1) is often used to simulate wave behaviour on the surface of shallow water. It has four fundamental Lie symmetries, which are outlined in Table 2 of this work. However, this study goes a step further by using approximation symmetry techniques to find other classes of the KdV equation that have more symmetries than the original, unperturbed form. Various exact perturbations to the KdV equation were used to identify the appropriate Lie symmetries. Notably, two major classes of perturbed KdV equations were found, each with five Lie symmetries. Tables 2, 4, and 5 show the symmetries and related conservation rules. In these tables, an additional symmetry was discovered, resulting in an additional conservation law, which reflects hidden information inside the system revealed by the perturbation process. This occurrence indicates that, while symmetry may not exist in the exact equation, perturbation can cause the equation to accept such symmetry. This impact is seen by comparing Tables 2–6. Table 1 identifies the determining PDEs that specify the collection of Lie symmetries for the given PDE. The precise PDE in Table 2 permits four Lie symmetries, however Tables 3 and 6 show just three, suggesting a loss of one symmetry (and one conservation law) in those circumstances.Tables 7 and 8 show four Lie symmetries, indicating that all conservation laws are restored in these cases. Finally, Table 9 summarises the Lie symmetries and their accompanying conservation laws, emphasising the complex link between symmetries and conserved values within the setting of the KdV equation.

6. Conflict of Interests

The author affirms that there is no conflict of interest regarding the publication of this paper.

References

[1] *Martin Braun and Martin Golubitsky.Differential equations and their applications, vol- ume 2.Springer, 1983.*

[2] *A Shamaoon, P Agarwal, C Cesarano, S Jain, et al. Approximate symmetries and conservation laws for mechanical systems described by mixed derivative perturbed pdes.ŽURNAL ÌNŽENERNIH NAUK, 10(2):8–15, 2023.*

[3] *HanzeLiu,JibinLi,andLeiLiu.Liesymmetryanalysis,optimalsystemsandexactsolutions tothefifth-orderkdvtypesofequations.JournalofMathematicalAnalysisandApplications, 368(2):551–558, 2010.*

[4] *Tamer A Abassy, Magdy A El-Tawil, and Hassan K Saleh.The solution of kdv and mkdv equations using adomian pade approximation. International Journal of Nonlinear Sciences and Numerical Simulation, 5(4):327–340, 2004.*

[5] *VitaliiAnvarovichBaikov,RafailKavyevichGazizov,andNailHairullovichIbragimov.Approximate symmetries.Matematicheskii Sbornik, 178(4):435–450, 1988.*

[6] *VABaikov,RKGazizov,andNKhIbragimov.Perturbationmethodsingroupanalysis. JournalofSovietMathematics,55:1450–1490,1991.*

[7] *Adnan Shamaoon, Zartab Ali, and Qaisar Maqbool. Conservation laws, exact solutions and nonlinear dispersion:A lie sym-metry approach.Journal of AppliedMath, 1(1):95, 2023.*

[8] *GraemeSegal.Thegeometryofthekdvequation.InternationalJournalofModernPhysicsA, 6(6):2859–2869, 1991.*

[9] *Kourosh Parand, Mehdi Dehghan, AR Rezaei, and SM Ghaderi.An approximation algorithmforthesolutionofthenonlinearlane–emdentypeequationsarisinginastrophysicsusinghermitefunctionscollocationmethod.ComputerPhysicsCommunications,181(6):1096–1108, 2010.*

[10] *George W Bluman.Applications of symmetry methods to partial differential equations. Springer, 2010.*

[11] *Adnan Shamaoon and Adeel Faruq.Analysis of symmetries, conservation laws, and exact solutions of (1+ 1) reaction-diffusion equation.International Journal of Science Academic Research, 05(07):7772–7783, 2024.*

[12] *HK Jassim, H Ahmad, A Shamaoon, and C Cesarano. An efficient hybrid technique for the solutionoffractional-orderpartialdifferentialequations.CarpathianMathematicalPublications, 13(3):790–804, 2021.*

[13] *Clemente Cesarano, William Ramírez, Stiven Díaz, AdnanShamaoon, andWaseemAhmad Khan.On apostol-type hermite degenerated polynomials.Mathematics, 11(8):1914, 2023.*

[14] *Xiao-Rui Hu,Sen-Yue Lou,and Yong Chen.Explicit solutions from eigenfunction symmetryofthekorteweg–devriesequation.PhysicalReviewE—Statistical,Nonlinear,andSoft Matter Physics, 85(5):056607, 2012*

[15] *Abdullahi Rashid Adem and Chaudry Masood Khalique.Symmetry reductions, exact solutions and conservation laws of a new coupled kdv system.Communications in Nonlinear Science and Numerical Simulation, 17(9):3465–3475, 2012.*

[16] *Martin Kruskal. Nonlinear wave equations. Dynamical Systems, Theory and Applications: Battelle Seattle 1974 Rencontres, pages 310–354, 2005.*

[17] *NiuXiaohuaandPanZuliang. Anewapproachtosolveperturbednonlinearevolutionequa- tions through lie-backlund symmetry method.Applied Mathematics-A Journal of Chinese Universities, 21:45–51, 2006.*

[18] *ZhangZhi-Yong,YongXue-Lin,andChenYu-Fu. Anewmethodtoobtainapproximatesym- metry of nonlinear evolution equation from perturbations.Chinese Physics B, 18(7):2629, 2009.*

[19] *Xiaoyu Jiao, Ruoxia Yao, and SY Lou.Approximate similarity reduction for singularly perturbed boussinesq equation via symmetry perturbation and direct method.Journal of Mathematical Physics, 49(9), 2008.*

[20] *Ashfaque H Bokhari, Abdul Hamid Kara, and FD Zaman.Invariant solutions of certain nonlinear evolution type equations with small parameters.Applied mathematics and computation, 182(2):1075–1082, 2006.*

[21] *DD Ganji and A Sadighi.Application of he's homotopy-perturbation method to nonlinear coupledsystemsofreaction-diffusionequations.InternationalJournalofNonlinearSciences and Numerical Simulation, 7(4):411–418, 2006.*

[22] *Ji-HuanHe.Addendum:newinterpretationofhomotopyperturbationmethod.Interna- tional journal of modern physics B, 20(18):2561–2568, 2006.*