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A Comparative Study of High Order Modified Second Derivative Simpson's Related Block Methods for Stiff Systems

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Abstract:

A continuous formulation based on the Modified Second Derivative Simpson's Block Methods (MSDSBM) with off-grid points are developed and adapted to cope with the integration of stiff systems of Ordinary Differential Equations (ODEs). The LU Decomposition technique was employed, which yields continuous formulation to derive the Standard Simpson's Block Method (SSBM), Second Derivative Simpson's Block Method (SDSBM), and the MSDSBM. This is achieved by combining the Modified Second Derivative Simpson's Method (MSDSM) with other additional methods (obtained from the same continuous formulation) and applying them as numerical integrators by assembling them into a single block matrix equation. The basic stability properties of the block methods was investigated and found to be zero-stable, consistent, and convergent, and from their regions of absolute stability, they possess regions suitable for the solution of all stiff ordinary differential equations. Further investigation showed that the newly constructed methods are all A-stable and of high order. The performance of the methods was demonstrated on some numerical examples to show accuracy and computational efficiency.

Keywords:

Collocation, Block Method, Higher Order, LU decomposition, Initial Value Problem.

1. INTRODUCTION

Differential equations were discovered when the need to understand the dynamic of systems became more demanding. They are found in sciences and engineering as well as economics social sciences, biology, business and health care. Many systems described by differential equations are so large and complex that a purely analytical solution is sometimes not traceable.

Numerical solutions for stiff systems of differential equations (ODEs) are very important in scientific computation, as they are widely used for solutions to real-world problems. In many applications modeled by systems of ordinary differential equations, these systems exhibit a behavior known as stiffness. Stiff systems are considered difficult because explicit numerical methods designed for non-stiff problems are used with very small step sizes or do not converge at all.

Stiffness is an important concept in the numerical solution of ordinary differential equations. It depends on the differential equation, the initial conditions, and the interval under consideration. Interest in stiff systems appeared initially in radio engineering (like the Van der Pol problem) at the beginning of the 20^{th} century. Then there was a new wave of interest in the middle 1950s with application in studying equations of chemical kinetics, and movement of celestial bodies, which contained both very slowly and very rapidly changing components.

2. Review of Related Literature

The first numerical method for differential equations was introduced by Euler in the 1760s and republished in his collected works in 1913. This method has order one and hence is not of any practical use, even though simple to implement. The first generalization of Euler's method was by Adams and Bashforth in 1883. Their methods used more information from the past to take a step forward. The Adams-Bashforth methods are a special case of a class of methods known as linear multistep methods, which take the form:

$$y_n = \alpha_1 y_{n+1} + \ldots + \alpha_k y_{n+k} + h[\beta_0 f(y_n) + \beta_1 f(y_{n+1}) + \ldots + \beta_k f(y_{n+k})]. \qquad (2.1)$$

In the case of the Adams-Bashforth methods, $\alpha_1 = 1, \alpha_2, \ldots, \alpha_k = 0,\ \beta_0 = 0.$ An extension of this idea was developed by Moulton in which $\beta_0 \neq 0$.This gives the method an implicit structure. Mohammed (2011) derived a linear multistep method with continuous coefficients and used it to obtain finite difference methods which were directly applied to solve first-order ODEs. Some methods, such as Taylor's series, numerical integration, and collocation method, two block methods for solving ordinary differential equations have been proposed by some researchers such as Abbas (1997) Mohammed et al (2010), Odekunle et al. (2012), Awari (2017), which does not require the development of starting values. A self-starting second derivative Top-order block method for the numerical integration of stiff systems of ODEs was obtained through the Rodriguez polynomial as a basis function, (Awari, Y.S., & Taparki, R., 2021).

A hybrid second derivative three-step method of order 7 was proposed for solving first-order stiff differential equations. The complementary and main methods were generated from a single continuous scheme through interpolation and collocation procedures. The continuous scheme makes it easy to interpolate at off-grid and grid points. The consistency, stability, and convergence properties of the block formula are presented. The hybrid second derivative block backward differentiation formula was concurrently applied to the first-order stiff systems to generate the numerical solution that does not coincide in time over a given interval, (Akinfenwa et al 2020).

3. METHODOLOGY

3.1 Derivation of Second Derivative Simpson's Method by LU Decomposition Method

$$v = (y_n, f_n, f_{n+1}, f_{n+2}, g_n, g_{n+1}, g_{n+2})$$

$$y(x) = \alpha_0(x)y_n + h\beta_0(x)f_n + h\beta_1(x)f_{n+1} + h\beta_2(x)f_{n+2} + h^2\gamma_0(x)g_n + h^2\gamma_1(x)g_{n+1} + h^2\gamma_2(x)g_{n+2} \quad (3.1)$$

$$\begin{aligned} y(x) = {} & (\alpha_{01} + \alpha_{02}x + \alpha_{03}x^2 + \alpha_{04}x^3 + \alpha_{05}x^4 + \alpha_{06}x^5 + \alpha_{07}x^6)y_n + \\ & (h\beta_{01} + h\beta_{02}x + h\beta_{03}x^2 + h\beta_{04}x^3 + h\beta_{05}x^4 + h\beta_{06}x^5 + h\beta_{07}x^6)f_n + \\ & (h\beta_{11} + h\beta_{12}x + h\beta_{13}x^2 + h\beta_{14}x^3 + h\beta_{15}x^4 + h\beta_{16}x^5 + h\beta_{17}x^6)f_{n+1} + \\ & (h\beta_{21} + h\beta_{22}x + h\beta_{23}x^2 + h\beta_{24}x^3 + h\beta_{25}x^4 + h\beta_{26}x^5 + h\beta_{27}x^6)f_{n+2} + \\ & (h^2\gamma_{01} + h^2\gamma_{02}x + h^2\gamma_{03}x^2 + h^2\gamma_{04}x^3 + h^2\gamma_{05}x^4 + h^2\gamma_{06}x^5 + h^2\gamma_{07}x^6)g_n + \\ & (h^2\gamma_{11} + h^2\gamma_{12}x + h^2\gamma_{13}x^2 + h^2\gamma_{14}x^3 + h^2\gamma_{15}x^4 + h^2\gamma_{16}x^5 + h^2\gamma_{17}x^6)g_{n+1} + \\ & (h^2\gamma_{21} + h^2\gamma_{22}x + h^2\gamma_{23}x^2 + h^2\gamma_{24}x^3 + h^2\gamma_{25}x^4 + h^2\gamma_{26}x^5 + h^2\gamma_{27}x^6)g_{n+2} \end{aligned}$$

then,

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 \\ 0 & 1 & 2x_n+2h & 3(x_n+h)^2 & 4(x_n+h)^3 & 5(x_n+h)^4 & 6(x_n+h)^5 \\ 0 & 1 & 2x_n+4h & 3(x_n+2h)^2 & 4(x_n+2h)^3 & 5(x_n+2h)^4 & 6(x_n+2h)^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_n+6h & 12(x_n+h)^2 & 20(x_n+h)^3 & 30(x_n+h)^4 \\ 0 & 0 & 2 & 6x_n+12h & 12(x_n+2h)^2 & 20(x_n+2h)^3 & 30(x_n+2h)^4 \end{bmatrix}$$

(3.2)

and the entries of L after applying LU Decomposition on Dbecomes

$l_{ij} = a_{ij} = l_{11} = 1,\ l_{21} = l_{31} = l_{41} = l_{51} = l_{61} = l_{71} = 0,$

$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj} = l_{22} = l_{32} = l_{42} = 1,\ l_{52} = l_{62} = l_{72} = 0,$

$l_{33} = 2h,\ l_{43} = 4h,\ l_{53} = 2,\ l_{63} = 2,\ l_{73} = 2,$

$l_{44} = 6h^2,\ l_{54} = -3h,\ l_{64} = 3h,\ l_{74} = 9h,$

$l_{55} = 8h^2,\ l_{65} = -4h^2,\ l_{75} = 8h^2,$

$l_{66} = -5h^3,\ l_{76} = 20h^3,$

$l_{77} = 24h^4.$

Hence we obtain the entries of c as

$$C=\Bigg[\bigg[c_{11}=1,\ c_{12}=-\frac{x_n\left(240\,h^5-460\,h^3\,x_n^2-495\,h^2\,x_n^3-204\,h\,x_n^4-30\,x_n^5\right)}{240\,h^5},\ c_{13}=-\frac{x_n^3\left(20\,h^2+15\,h\,x_n+3\,x_n^2\right)}{15\,h^4},\ c_{14}=-\frac{x_n^3\left(140\,h^3+255\,h^2\,x_n+156\,h\,x_n^2+30\,x_n^3\right)}{240\,h^5},\ c_{15}=\frac{x_n^2\left(120\,h^4+240\,h^3\,x_n+195\,h^2\,x_n^2+72\,h\,x_n^3+10\,x_n^4\right)}{240\,h^4},\ c_{16}=\frac{x_n^3\left(8\,h^3+12\,h^2\,x_n+6\,h\,x_n^2+x_n^3\right)}{6\,h^4},\ c_{17}=\frac{x_n^3\left(40\,h^3+75\,h^2\,x_n+48\,h\,x_n^2+10\,x_n^3\right)}{240\,h^4}\bigg],$$

$$\bigg[c_{21}=0,\ c_{22}=\frac{4\,h^5-23\,h^3\,x_n^2-33\,h^2\,x_n^3-17\,h\,x_n^4-3\,x_n^5}{4\,h^5},\ c_{23}=\frac{x_n^2\left(4\,h^2+4\,h\,x_n+x_n^2\right)}{h^4},\ c_{24}=\frac{x_n^2\left(7\,h^3+17\,h^2\,x_n+13\,h\,x_n^2+3\,x_n^3\right)}{4\,h^5},\ c_{25}=-\frac{x_n\left(4\,h^4+12\,h^3\,x_n+13\,h^2\,x_n^2+6\,h\,x_n^3+x_n^4\right)}{4\,h^4},\ c_{26}=-\frac{x_n^2\left(4\,h^3+8\,h^2\,x_n+5\,h\,x_n^2+x_n^3\right)}{h^4},\ c_{27}=-\frac{x_n^2\left(2\,h^3+5\,h^2\,x_n+4\,h\,x_n^2+x_n^3\right)}{4\,h^4}\bigg],$$

$$\bigg[c_{31}=0,\ c_{32}=\frac{x_n\left(46\,h^3+99\,h^2\,x_n+68\,h\,x_n^2+15\,x_n^3\right)}{8\,h^5},\ c_{33}=-\frac{2\,x_n\left(2\,h^2+3\,h\,x_n+x_n^2\right)}{h^4},\ c_{34}=-\frac{x_n\left(14\,h^3+51\,h^2\,x_n+52\,h\,x_n^2+15\,x_n^3\right)}{8\,h^5},\ c_{35}=\frac{4\,h^4+24\,h^3\,x_n+39\,h^2\,x_n^2+24\,h\,x_n^3+5\,x_n^4}{8\,h^4},\ c_{36}=\frac{x_n\left(8\,h^3+24\,h^2\,x_n+20\,h\,x_n^2+5\,x_n^3\right)}{2\,h^4},\ c_{37}=\frac{x_n\left(4\,h^3+15\,h^2\,x_n+16\,h\,x_n^2+5\,x_n^3\right)}{8\,h^4}\bigg],$$

$$\bigg[c_{41}=0,\ c_{42}=-\frac{23\,h^3+99\,h^2\,x_n+102\,h\,x_n^2+30\,x_n^3}{12\,h^5},\ c_{43}=\frac{4\,h^2+12\,h\,x_n+6\,x_n^2}{3\,h^4},\ c_{44}=\frac{7\,h^3+51\,h^2\,x_n+78\,h\,x_n^2+30\,x_n^3}{12\,h^5},\ c_{45}=-\frac{12\,h^3+39\,h^2\,x_n+36\,h\,x_n^2+10\,x_n^3}{12\,h^4},\ c_{46}=-\frac{4\,h^3+24\,h^2\,x_n+30\,h\,x_n^2+10\,x_n^3}{3\,h^4},\ c_{47}=-\frac{2\,h^3+15\,h^2\,x_n+24\,h\,x_n^2+10\,x_n^3}{12\,h^4}\bigg],$$

$$\bigg[c_{51}=0,\ c_{52}=\frac{33\,h^2+68\,h\,x_n+30\,x_n^2}{16\,h^5},\ c_{53}=-\frac{x_n+h}{h^4},\ c_{54}=-\frac{17\,h^2+52\,h\,x_n+30\,x_n^2}{16\,h^5},\ c_{55}=\frac{13\,h^2+24\,h\,x_n+10\,x_n^2}{16\,h^4},\ c_{56}=\frac{4\,h^2+10\,h\,x_n+5\,x_n^2}{2\,h^4},\ c_{57}=\frac{5\,h^2+16\,h\,x_n+10\,x_n^2}{16\,h^4}\bigg],$$

$$\bigg[c_{61}=0,\ c_{62}=-\frac{17\,h+15\,x_n}{20\,h^5},\ c_{63}=\frac{1}{5\,h^4},\ c_{64}=\frac{13\,h+15\,x_n}{20\,h^5},\ c_{65}=-\frac{6\,h+5\,x_n}{20\,h^4},\ c_{66}=-\frac{x_n+h}{h^4},\ c_{67}=-\frac{4\,h+5\,x_n}{20\,h^4}\bigg],$$

$$\bigg[c_{71}=0,\ c_{72}=\frac{1}{8\,h^5},\ c_{73}=0,\ c_{74}=-\frac{1}{8\,h^5},\ c_{75}=\frac{1}{24\,h^4},\ c_{76}=\frac{1}{6\,h^4},\ c_{77}=\frac{1}{24\,h^4}\bigg]\Bigg]$$

Substituting the values of c as in sub-section (2.2), yield the continuous formulation for the Second Derivative Simpson's Method as:

$$y(\xi)=y_n+\left(\xi-\frac{23}{12}\frac{\xi^3}{h^2}+\frac{33}{16}\frac{\xi^4}{h^3}-\frac{17}{20}\frac{\xi^5}{h^4}+\frac{1}{8}\frac{\xi^6}{h^5}\right)f_n+\left(\frac{4}{3}\frac{\xi^3}{h^2}-\frac{\xi^4}{h^3}+\frac{1}{5}\frac{\xi^5}{h^4}\right)f_{n+1}+\left(\frac{7}{12}\frac{\xi^3}{h^2}-\frac{17}{16}\frac{\xi^4}{h^3}+\frac{13}{20}\frac{\xi^5}{h^4}-\frac{1}{8}\frac{\xi^6}{h^5}\right)f_{n+2}+\left(\frac{1}{2}\xi^2-\frac{\xi^3}{h}+\frac{13}{16}\frac{\xi^4}{h^2}-\frac{3}{10}\frac{\xi^5}{h^3}+\frac{1}{24}\frac{\xi^6}{h^4}\right)g_n+\left(-\frac{4}{3}\frac{\xi^3}{h}+\frac{2\,\xi^4}{h^2}-\frac{\xi^5}{h^3}+\frac{1}{6}\frac{\xi^6}{h^4}\right)g_{n+1}+\left(-\frac{1}{6}\frac{\xi^3}{h}+\frac{5}{16}\frac{\xi^4}{h^2}-\frac{1}{5}\frac{\xi^5}{h^3}+\frac{1}{24}\frac{\xi^6}{h^4}\right)g_{n+2}$$

(3.3)

where $g_i = f_i' = y_i''\quad x_n \le x \le x_{n+2}$,

Evaluating (3.3) at $x = x_{n+2}$ derive

$$y_{n+2} - y_n = \frac{h}{15}\left[7f_n + 16f_{n+1} + 7f_{n+2}\right] + \frac{h^2}{15}\left[g_n - g_{n+2}\right],$$

(3.4)

Equation (3.4) has order $p = 6$, and $c_7 = \frac{1}{4725}$

Equation (3.4) is the Second Derivative Simpson's Method.

3.1.1 Derivation of Second Derivative Simpson's Block Method from the Continuous Formulation

Substituting $x = x_{n+1}$ in equation (3.3) gives

$$y_{n+1} - y_n = \frac{h}{240}\left[101f_n + 128f_{n+1} + 11f_{n+2}\right] + \frac{h^2}{240}\left[13g_n - 40g_{n+1} - 3g_{n+2}\right].$$

(3.5)

Equation (3.5) has order $p = 6$, and $c_7 = \frac{1}{9450}$

Now, (3.4) & (3.5) can be put together as follows;

$$\left\{\begin{aligned} y_{n+2} - y_n &= \frac{h}{15}\left[7f_n + 16f_{n+1} + 7f_{n+2}\right] + \frac{h^2}{15}\left[g_n - g_{n+2}\right] \\ y_{n+1} - y_n &= \frac{h}{240}\left[101f_n + 128f_{n+1} + 11f_{n+2}\right] + \frac{h^2}{240}\left[13g_n - 40g_{n+1} - 3g_{n+2}\right] \end{aligned}\right\} \quad (3.6)$$

3.1.2 Zero Stability of the Second Derivative Simpson's Block Method (SDSBM)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + h\begin{bmatrix} \frac{8}{15} & \frac{11}{240} \\ \frac{16}{15} & \frac{7}{15} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{101}{240} \\ 0 & \frac{7}{15} \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} + h^2\begin{bmatrix} -\frac{1}{6} & -\frac{1}{80} \\ 0 & -\frac{1}{15} \end{bmatrix}$$

$$\begin{bmatrix} g_{n+1} \\ g_{n+2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{13}{240} \\ 0 & \frac{1}{15} \end{bmatrix}\begin{bmatrix} g_{n-1} \\ g_n \end{bmatrix},$$

$$p(\lambda)=\det[\lambda I-A]=0,\quad p(\lambda)=\det\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}-\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}=\det\begin{bmatrix}\lambda & -1\\ 0 & \lambda-1\end{bmatrix}=\lambda(\lambda-1)=0,\qquad \lambda_1=0,\ \lambda_2=1.$$

The second derivative Simpson's method is zero (0) stable, and the method is also convergence because is consistence and zero stable.

3.1.3 Region of Absolute Stability of the Second Derivative Simpson's Block Method

The stability properties of the Second Derivative Simpson's Block Method (SDSBM), has reformulated (3.6) as general linear methods. In which a general linear method is represented by a partitioned $(s+r)\times(s+r)$ matrix, (containing A, U, B, and V), and using MATLAB code to generate the region of absolute stability of SDSBM

$$\begin{bmatrix}Y^{[n]}\\ y^{[n-1]}\end{bmatrix}=\left[\begin{array}{c|c}A & U\\ \hline B & V\end{array}\right]\begin{bmatrix}hf\ (Y^{[n]})\\ y^{[n]}\end{bmatrix},\qquad n=1,2,\ldots,N,$$

where,

$$Y^{[n]}=\begin{bmatrix}Y_1^{[n]}\\ Y_2^{[n]}\\ \vdots\\ Y_s^{[n]}\end{bmatrix},\qquad Y^{[n-1]}=\begin{bmatrix}Y_1^{[n-1]}\\ Y_2^{[n-1]}\\ \vdots\\ Y_r^{[n-1]}\end{bmatrix}$$

and

$$A=\begin{bmatrix}0 & 0 & 0\\ \frac{101}{240} & \frac{16}{30} & \frac{11}{240}\\ \frac{7}{15} & \frac{16}{15} & \frac{7}{15}\end{bmatrix},\ U=\begin{bmatrix}0 & 1\\ 0 & 1\\ 0 & 1\end{bmatrix},\ B=\begin{bmatrix}\frac{7}{15} & \frac{16}{15} & \frac{7}{15}\\ \frac{101}{240} & \frac{16}{30} & \frac{11}{240}\end{bmatrix},\ V=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}.$$

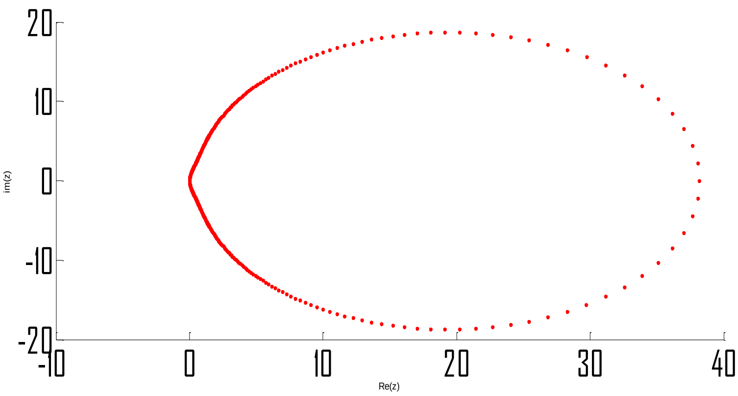

Fig. 1: Region of Absolute Stability of the SDSBM

3.2 Derivation of the Modified Second Derivative Simpson's Method (MSDSM) with off-step Collocation Points at $x_{n+\frac{1}{2}}$

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_n + h & 3\left(x_n + \frac{1}{2}h\right)^2 & 4\left(x_n + \frac{1}{2}h\right)^3 & 5\left(x_n + \frac{1}{2}h\right)^4 & 6\left(x_n + \frac{1}{2}h\right)^5 & 7\left(x_n + \frac{1}{2}h\right)^6 \\ 0 & 1 & 2x_n + 2h & 3(x_n + h)^2 & 4(x_n + h)^3 & 5(x_n + h)^4 & 6(x_n + h)^5 & 7(x_n + h)^6 \\ 0 & 1 & 2x_n + 4h & 3(x_n + 2h)^2 & 4(x_n + 2h)^3 & 5(x_n + 2h)^4 & 6(x_n + 2h)^5 & 7(x_n + 2h)^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\ 0 & 0 & 2 & 6x_n + 6h & 12(x_n + h)^2 & 20(x_n + h)^3 & 30(x_n + h)^4 & 42(x_n + h)^5 \\ 0 & 0 & 2 & 6x_n + 12h & 12(x_n + 2h)^2 & 20(x_n + 2h)^3 & 30(x_n + 2h)^4 & 42(x_n + 2h)^5 \end{bmatrix} \quad (3.7)$$

And the entries of c are listed as follows:

$$C=\Bigg[\Bigg[c_{11}=1,\ c_{12}=-\frac{x_n\left(560h^6-2940h^4x_n^2-5355h^3x_n^3-4116h^2x_n^4-1470hx_n^5-200x_n^6\right)}{560h^6},\ c_{13}=-\frac{64x_n^3\left(140h^4+315h^3x_n+273h^2x_n^2+105hx_n^3+15x_n^4\right)}{945h^6},$$

$$=\frac{x_n^3\left(140h^4+385h^3x_n+357h^2x_n^2+140hx_n^3+20x_n^4\right)}{35h^6},\ c_{15}=\frac{x_n^3\left(3500h^4+11655h^3x_n+14196h^2x_n^2+7350hx_n^3+1320x_n^4\right)}{15120h^6},\ c_{16}$$

$$=\frac{x_n^2\left(840h^5+2800h^4x_n+3885h^3x_n^2+2688h^2x_n^3+910hx_n^4+120x_n^5\right)}{1680h^5},\ c_{17}=-\frac{x_n^3\left(280h^4+840h^3x_n+882h^2x_n^2+385hx_n^3+60x_n^4\right)}{210h^5},\ c_{18}=$$

$$-\frac{x_n^3\left(280h^4+945h^3x_n+1176h^2x_n^2+630hx_n^3+120x_n^4\right)}{5040h^5}\Bigg],$$

$$\Bigg[c_{21}=0,\ c_{22}=\frac{4h^6-63h^4x_n^2-153h^3x_n^3-147h^2x_n^4-63hx_n^5-10x_n^6}{4h^6},\ c_{23}=\frac{64x_n^2\left(4h^4+12h^3x_n+13h^2x_n^2+6hx_n^3+x_n^4\right)}{9h^6},\ c_{24}=$$

$$-\frac{x_n^2\left(12h^4+44h^3x_n+51h^2x_n^2+24hx_n^3+4x_n^4\right)}{h^6},\ c_{25}=-\frac{x_n^2\left(25h^4+111h^3x_n+169h^2x_n^2+105hx_n^3+22x_n^4\right)}{36h^6},\ c_{26}=$$

$$-\frac{x_n\left(4h^5+20h^4x_n+37h^3x_n^2+32h^2x_n^3+13hx_n^4+2x_n^5\right)}{4h^5},\ c_{27}=\frac{x_n^2\left(4h^4+16h^3x_n+21h^2x_n^2+11hx_n^3+2x_n^4\right)}{h^5},\ c_{28}$$

$$=\frac{x_n^2\left(2h^4+9h^3x_n+14h^2x_n^2+9hx_n^3+2x_n^4\right)}{12h^5}\Bigg],$$

$$\Bigg[c_{31}=0,\ c_{32}=\frac{3x_n\left(42h^4+153h^3x_n+196h^2x_n^2+105hx_n^3+20x_n^4\right)}{8h^6},\ c_{33}=-\frac{64x_n\left(4h^4+18h^3x_n+26h^2x_n^2+15hx_n^3+3x_n^4\right)}{9h^6},\ c_{34}$$

$$=\frac{6x_n\left(2h^4+11h^3x_n+17h^2x_n^2+10hx_n^3+2x_n^4\right)}{h^6},\ c_{35}=\frac{x_n\left(50h^4+333h^3x_n+676h^2x_n^2+525hx_n^3+132x_n^4\right)}{72h^6},\ c_{36}$$

$$=\frac{4h^5+40h^4x_n+111h^3x_n^2+128h^2x_n^3+65hx_n^4+12x_n^5}{8h^5},\ c_{37}=-\frac{x_n\left(8h^4+48h^3x_n+84h^2x_n^2+55hx_n^3+12x_n^4\right)}{2h^5},\ c_{38}=$$

$$-\frac{x_n\left(4h^4+27h^3x_n+56h^2x_n^2+45hx_n^3+12x_n^4\right)}{24h^5}\Bigg],$$

$$\Bigg[c_{41}=0,\ c_{42}=-\frac{21h^4+153h^3x_n+294h^2x_n^2+210hx_n^3+50x_n^4}{4h^6},\ c_{43}=\frac{64\left(4h^4+36h^3x_n+78h^2x_n^2+60hx_n^3+15x_n^4\right)}{27h^6},\ c_{44}=$$

$$-\frac{2\left(2h^4+22h^3x_n+51h^2x_n^2+40hx_n^3+10x_n^4\right)}{h^6},\ c_{45}=-\frac{25h^4+333h^3x_n+1014h^2x_n^2+1050hx_n^3+330x_n^4}{108h^6},\ c_{46}=$$

$$-\frac{20h^4+111h^3x_n+192h^2x_n^2+130hx_n^3+30x_n^4}{12h^5},\ c_{47}=\frac{2\left(2h^4+24h^3x_n+63h^2x_n^2+55hx_n^3+15x_n^4\right)}{3h^5},\ c_{48}=\frac{2h^4+27h^3x_n+84h^2x_n^2+90hx_n^3+30x_n^4}{36h^5}\Bigg],$$

$$\Bigg[c_{51}=0,\ c_{52}=\frac{153h^3+588h^2x_n+630hx_n^2+200x_n^3}{16h^6},\ c_{53}=-\frac{64\left(3h^3+13h^2x_n+15hx_n^2+5x_n^3\right)}{9h^6},\ c_{54}=\frac{11h^3+51h^2x_n+60hx_n^2+20x_n^3}{h^6},\ c_{55}$$

$$=\frac{111h^3+676h^2x_n+1050hx_n^2+440x_n^3}{144h^6},\ c_{56}=\frac{37h^3+128h^2x_n+130hx_n^2+40x_n^3}{16h^5},\ c_{57}=-\frac{8h^3+42h^2x_n+55hx_n^2+20x_n^3}{2h^5},\ c_{58}=$$

$$-\frac{9h^3+56h^2x_n+90hx_n^2+40x_n^3}{48h^5}\Bigg],$$

$$\Bigg[c_{61}=0,\ c_{62}=-\frac{3\left(49h^2+105hx_n+50x_n^2\right)}{20h^6},\ c_{63}=\frac{64\left(13h^2+30hx_n+15x_n^2\right)}{45h^6},\ c_{64}=-\frac{3\left(17h^2+40hx_n+20x_n^2\right)}{5h^6},\ c_{65}=-\frac{169h^2+525hx_n+330x_n^2}{180h^6},\ c_{66}=$$

$$-\frac{32h^2+65hx_n+30x_n^2}{20h^5},\ c_{67}=\frac{21h^2+55hx_n+30x_n^2}{5h^5},\ c_{68}=\frac{14h^2+45hx_n+30x_n^2}{60h^5}\Bigg],$$

$$\Bigg[c_{71}=0,\ c_{72}=\frac{21h+20x_n}{8h^6},\ c_{73}=-\frac{64\left(x_n+h\right)}{9h^6},\ c_{74}=\frac{4\left(x_n+h\right)}{h^6},\ c_{75}=\frac{35h+44x_n}{72h^6},\ c_{76}=\frac{13h+12x_n}{24h^5},\ c_{77}=-\frac{11h+12x_n}{6h^5},\ c_{78}=-\frac{3h+4x_n}{24h^5}\Bigg],$$

$$\Bigg[c_{81}=0,\ c_{82}=-\frac{5}{14h^6},\ c_{83}=\frac{64}{63h^6},\ c_{84}=-\frac{4}{7h^6},\ c_{85}=-\frac{11}{126h^6},\ c_{86}=-\frac{1}{14h^5},\ c_{87}=\frac{2}{7h^5},\ c_{88}=\frac{1}{42h^5}\Bigg]\Bigg]$$

The continuous formulation of the Modified Second Derivative Simpson's Method gives;

$$y(\xi) = y_n + \left(\xi - \frac{21}{4}\frac{\xi^3}{h^2} + \frac{153}{16}\frac{\xi^4}{h^3} - \frac{147}{20}\frac{\xi^5}{h^4} + \frac{21}{8}\frac{\xi^6}{h^5} - \frac{5}{14}\frac{\xi^7}{h^6}\right) f_n + \left(\frac{256}{27}\frac{\xi^3}{h^2} - \frac{64}{3}\frac{\xi^4}{h^3} + \frac{832}{45}\frac{\xi^5}{h^4} - \frac{64}{9}\frac{\xi^6}{h^5} + \frac{64}{63}\frac{\xi^7}{h^6}\right) f_{n+\frac{1}{2}} + \left(-\frac{4\xi^3}{h^2} + \frac{11\xi^4}{h^3} - \frac{51}{5}\frac{\xi^5}{h^4} + \frac{4\xi^6}{h^5} - \frac{4}{7}\frac{\xi^7}{h^6}\right) f_{n+1} + \left(-\frac{25}{108}\frac{\xi^3}{h^2} + \frac{37}{48}\frac{\xi^4}{h^3} - \frac{169}{180}\frac{\xi^5}{h^4} + \frac{35}{72}\frac{\xi^6}{h^5} - \frac{11}{126}\frac{\xi^7}{h^6}\right) f_{n+2} + \left(\frac{1}{2}\xi^2 - \frac{5}{3}\frac{\xi^3}{h} + \frac{37}{16}\frac{\xi^4}{h^2} - \frac{8}{5}\frac{\xi^5}{h^3} + \frac{13}{24}\frac{\xi^6}{h^4} - \frac{1}{14}\frac{\xi^7}{h^5}\right) g_n + \left(\frac{4}{3}\frac{\xi^3}{h} - \frac{4\xi^4}{h^2} + \frac{21}{5}\frac{\xi^5}{h^3} - \frac{11}{6}\frac{\xi^6}{h^4} + \frac{2}{7}\frac{\xi^7}{h^5}\right) g_{n+1} + \left(\frac{1}{18}\frac{\xi^3}{h} - \frac{3}{16}\frac{\xi^4}{h^2} + \frac{7}{30}\frac{\xi^5}{h^3} - \frac{1}{8}\frac{\xi^6}{h^4} + \frac{1}{42}\frac{\xi^7}{h^5}\right) g_{n+2}$$

(3.8)

Evaluating (3.8) at $x = x_{n+1}$, $x = x_{n+\frac{1}{2}}$ and $x = x_{n+2}$ provide the following discrete schemes accordingly

$$y_{n+2} - y_n = \frac{h}{945}\left[353 f_{n+2} + 432 f_{n+1} + 1024 f_{n+\frac{1}{2}} + 81 f_n\right] - \frac{h^2}{315}\left[13 g_{n+2} - 96 g_{n+1} + 3 g_n\right],$$

(3.9)

Equation (3.9) has order $p = 7$, and $c_8 = \frac{1}{66150}$

$$y_{n+1} - y_n = \frac{h}{15120}\left[3483 f_n + 8192 f_{n+\frac{1}{2}} + 3456 f_{n+1} - 11 f_{n+2}\right] + \frac{h^2}{5040}\left[g_{n+2} - 72 g_{n+1} + 81 g_n\right], \quad (3.10)$$

Equation (3.10) has order $p = 7$, and $c_8 = \frac{1}{1411200}$

$$y_{n+\frac{1}{2}} - y_n = \frac{h}{483840}\left[120933 f_n + 157952 f_{n+\frac{1}{2}} - 354246 f_{n+1} - 1541 f_{n+2}\right] + \frac{h^2}{16128}\left[121 g_{n+2} + 3468 g_{n+1} + 3081 g_n\right]$$

(3.11)

Equation (3.11) has order $p = 7$, and $c_8 = \frac{1679}{1083801600}$

The block form of the method is;

$$\begin{cases} y_{n+2}-y_n=\dfrac{h}{945}\left[353f_{n+2}+432f_{n+1}+1024f_{n+\frac{1}{2}}+81f_n\right]-\dfrac{h^2}{315}\left[13g_{n+2}-96g_{n+1}+3g_n\right] \\ y_{n+1}-y_n=\dfrac{h}{15120}\left[3483f_n+8192f_{n+\frac{1}{2}}+3456f_{n+1}-11f_{n+2}\right]+\dfrac{h^2}{5040}\left[g_{n+2}-72g_{n+1}+81g_n\right] \\ y_{n+\frac{1}{2}}-y_n=\dfrac{h}{483840}\left[120933f_n+157952f_{n+\frac{1}{2}}-354246f_{n+1}-1541f_{n+2}\right]+\dfrac{h^2}{16128} \\ \left[121g_{n+2}+3468g_{n+1}+3081g_n\right] \end{cases} \quad (3.12)$$

Equation (3.12) is called the Modified Second Derivative Simpson's Block Method (MSDSBM).

3.2.1 Zero Stability of the Modified Second Derivative Simpson's Block Method with off-step Collocation Points at $x_{n+\frac{1}{2}}$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_{n-\frac{1}{2}}\\y_n\end{bmatrix}+h\begin{bmatrix}\frac{617}{1890}&-\frac{41}{560}&-\frac{1541}{483840}\\\frac{512}{945}&\frac{8}{35}&-\frac{11}{15120}\\\frac{1024}{945}&\frac{16}{35}&\frac{353}{945}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+2}\end{bmatrix}$$

$$+\begin{bmatrix}0&0&\frac{4479}{17920}\\0&0&\frac{129}{560}\\0&0&\frac{3}{35}\end{bmatrix}\begin{bmatrix}f_{n-1}\\f_{n-\frac{1}{2}}\\f_n\end{bmatrix}+h^2\begin{bmatrix}0&\frac{298}{13440}&\frac{121}{161280}\\0&-\frac{1}{70}&\frac{1}{5040}\\0&\frac{32}{105}&-\frac{13}{315}\end{bmatrix}\begin{bmatrix}g_{n+\frac{1}{2}}\\g_{n+1}\\g_{n+2}\end{bmatrix}+\begin{bmatrix}0&0&\frac{1027}{53760}\\0&0&\frac{9}{560}\\0&0&-\frac{1}{105}\end{bmatrix}\begin{bmatrix}g_{n-1}\\g_{n-\frac{1}{2}}\\g_n\end{bmatrix}$$

$$p(\lambda)=\det[\lambda I-A]=0,\quad p(\lambda)=\det\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}-\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}=\det\begin{bmatrix}\lambda&0&-1\\0&\lambda&-1\\0&0&\lambda-1\end{bmatrix}=\lambda^2(\lambda-1)=0$$

$$where\quad \lambda_1=\lambda_2=0,\ \lambda_3=1$$

The modified second derivative Simpson's block method of collocation points at $x_{n+\frac{1}{2}}$ is zero (0) stable, the method is also convergent because it has order > 1.

3.2.2 Region of Absolute Stability of the Modified Second Derivative Simpson's Block Method with off-step Collocation Points at $x_{n+\frac{1}{2}}$

The stability properties of the Modified Second Derivative Simpson's Block Method (MSDSBM), has reformulated (3.12) as general linear methods. In which a general linear method is represented by a partitioned $(s+r)\times(s+r)$ matrix, (containing A, U, B, and V),

$$\begin{bmatrix} Y^{[n]} \\ y^{[n-1]} \end{bmatrix} = \begin{bmatrix} \frac{A}{B} \frac{U}{V} \end{bmatrix} \begin{bmatrix} hf\ (Y^{[n]}) \\ y^{[n]} \end{bmatrix}, \qquad n = 1, 2, \ldots, N,$$

where,

$$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \vdots \\ Y_s^{[n]} \end{bmatrix}, \quad Y^{[n-1]} = \begin{bmatrix} Y_1^{[n-1]} \\ Y_2^{[n-1]} \\ \vdots \\ Y_r^{[n-1]} \end{bmatrix}, \quad f(Y^{[n]}) = \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ \vdots \\ f(Y_s^{[n]}) \end{bmatrix}, \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix} \text{ and}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{4479}{17920} & \frac{617}{1890} & -\frac{41}{560} & -\frac{1541}{483840} \\ \frac{129}{560} & \frac{512}{945} & \frac{8}{35} & -\frac{11}{15120} \\ \frac{3}{35} & \frac{1024}{945} & \frac{16}{35} & \frac{353}{945} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{3}{35} & \frac{1024}{945} & \frac{16}{35} & \frac{353}{945} \\ \frac{129}{560} & \frac{512}{945} & \frac{8}{35} & -\frac{11}{15120} \end{bmatrix},$$

$$V = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

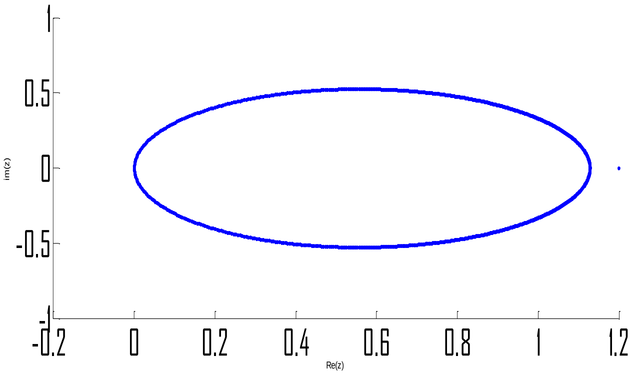

Figure 2: Region of Absolute Stability of the MSDSBM

The Modified Second Derivative Simpson's Block Methods (MSDSBM) has uniform and higher order $p=7$, hence increasing the accuracy of the method for the same step number $k=2$.The regions of absolute stability of the methods (3.6), and (3.12) is A-stable, since the regions consists of complex plane outside the enclosed figure.

4. RESULTS

4.1 Numerical Experiments

Problem 4.1.1:**Consider a** linear Stiff system in three-dimensions on the interval $0 \le x \le 1$

$$\begin{aligned} y'_1 &= -21y_1 + 19y_2 - 20y_3, \quad y_1(0)=1, \\ y'_2 &= 19y_1 - 21y_2 + 20y_3, \quad y_2(0)=0, \\ y'_3 &= 40y_1 - 40y_2 - 40y_3, \quad y_3(0)=-1. \end{aligned}$$

The analytical solution of the system is given by

$$y_1(x) = \frac{1}{2}(e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x))),$$

$$y_2(x) = \frac{1}{2}(e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x))),$$

$$y_3(x) = \frac{1}{2}(2e^{-40x}(\cos(40x) - \sin(40x))).$$

The problem was solved using the derived block methods and the results obtained are presented in Tables 1 and 2, while the solution curves are displayed in Figures 3 and 4.

Table 1: Comparison of Analytical and Approximate Solution for Problem 4.1.1

N	Analytical Sol. y'	SDSBM y'	MSDSBM y'
20	6.76676×10^{-2}	6.76676×10^{-2}	6.76676×10^{-2}
40	6.76676×10^{-2}	6.76676×10^{-2}	6.76676×10^{-2}
80	6.76676×10^{-2}	6.76676×10^{-2}	6.76676×10^{-2}
160	6.76676×10^{-2}	6.76676×10^{-2}	6.76676×10^{-2}
320	6.76676×10^{-2}	6.76676×10^{-2}	6.76676×10^{-2}
640	6.76676×10^{-2}	6.76676×10^{-2}	6.76676×10^{-2}

Table 2: Computation of Absolute Errors for Problem 4.1.1

N	SDSBM $p=6$	MSDSBD $p=7$	SSDM $p=7$	SDGAM $p=6$	SDGAM $p=8$
20	1.8×10^{-11}	1.2×10^{-14}	2.9×10^{-3}	1.3×10^{-11}	7.3×10^{-15}
40	2.5×10^{-13}	8.3×10^{-17}	6.8×10^{-5}	2.1×10^{-13}	1.3×10^{-17}
80	3.8×10^{-15}	0.0	1.8×10^{-6}	3.2×10^{-15}	1.3×10^{-17}
160	8.3×10^{-17}	1.4×10^{-17}	2.9×10^{-8}	6.5×10^{-17}	0.0
320	2.8×10^{-17}	4.2×10^{-17}	4.6×10^{-10}	1.3×10^{-17}	2.6×10^{-17}
640	5.6×10^{-17}	5.6×10^{-17}	7.4×10^{-12}	2.6×10^{-17}	0.0

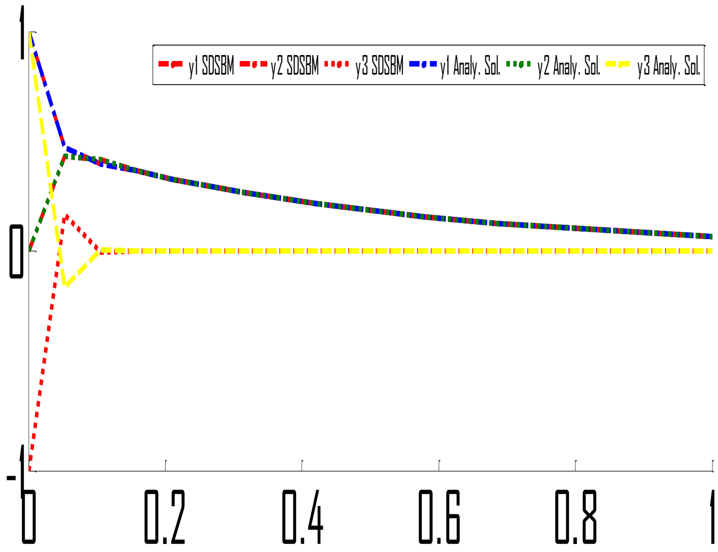

Fig. 3: Comparison of SDSBM with Analytical Solution

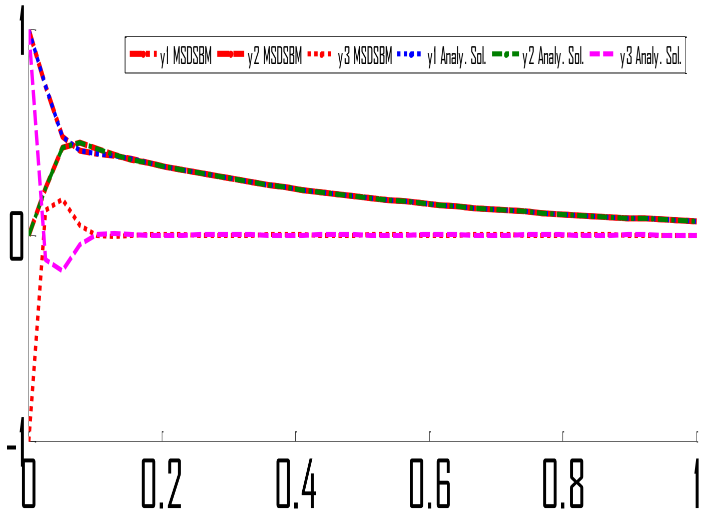

Fig. 4: Comparison of MSDSBM with Analytical Solution

Problem 4.1.2: **Given a Stiff Initial Value Problem (IVP) of** the form

$$\begin{aligned} y' &= -y_1 + 95y_2, \quad y(0) = 1, \\ z' &= -y - 97z, \quad z(0) = 1. \end{aligned} \text{, with } 0 \le x \le 1,$$

The Analytical solution given as:

$$\begin{aligned} y(x) &= \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x}, \\ z(x) &= \frac{48}{47}e^{-96x} - \frac{1}{47}e^{-2x}. \end{aligned}$$

Table 3: Comparison of Analytical and Approximate Solution for Problem 4.1.2

N	Analytical Solution y'_n	SDSBM y'	MSDSBM y'
0.0625	2.73550×10^{-1}	2.73550×10^{-1}	2.73550×10^{-1}
0.03125	2.73550×10^{-1}	2.73550×10^{-1}	2.73550×10^{-1}

Table 4: Computation of Absolute Errors for Problem 4.1.2

N	SDSBM	MSDSBM	Sahi R.K.		
	y'	y'	SSDM	Jackson	Cash
0.0625	3.02864×10^{-10}	2.46414×10^{-13}	9×10^{-11}	3×10^{-7}	3×10^{-7}
0.03125	4.03161×10^{-12}	1.55431×10^{-15}	4×10^{-12}	1×10^{-8}	1×10^{-8}

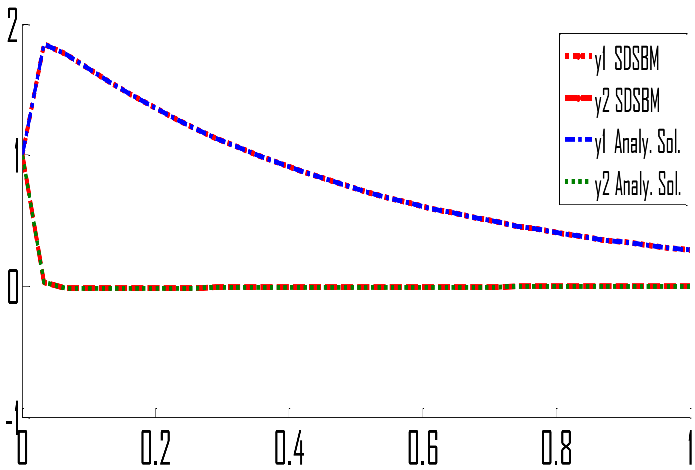

Fig. 5: Comparison of SDSBM with Analytical Solution

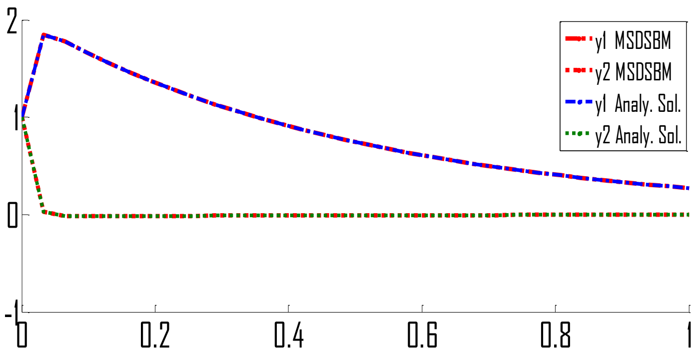

Fig. 6: Comparison of MSDSBM with Analytical Solution

Problem 4.1.3: The third test problem a Stiff system with interval $0 \le x \le 10$, given by

$$\begin{bmatrix} y'_1(x) \\ y'_2(x) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 998 & -999 \end{bmatrix} + \begin{bmatrix} 2\sin x \\ 999(\cos x - \sin x) \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

with the analytical solution as,

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} \sin x + 2e^{-x} \\ \cos x + 2e^{-x} \end{bmatrix}$$

Table 5: Computation of Absolute Errors for Problem 4.1.3

H	SDSBM	MSDSBM	NEHPS	EHPS
0.05	9.28113×10^{-4}	5.85374×10^{-3}	2.2144×10^{-3}	1.6499×10^{-6}
0.01	2.06163×10^{-4}	1.35240×10^{-3}	6.9207×10^{-5}	5.2834×10^{-7}
0.005	1.00525×10^{-4}	6.87871×10^{-4}	1.3166×10^{-5}	9.9938×10^{-9}
0.001	2.15029×10^{-5}	1.39521×10^{-4}	8.5838×10^{-6}	6.4851×10^{-10}

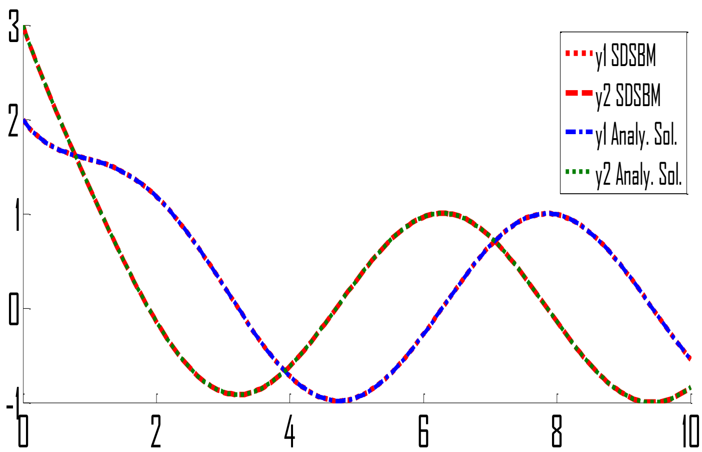

Fig. 7: Comparison of SDSBM with Analytical Solution

4.2 Discussion of Results

The derivation of the new methods schemes from the same continuous formulation and was effectively implemented in block forms for the numerical integration of stiff system of ordinary differential equations. Stability analysis of the block method showed zero stability and high uniform order p = 7 in MSDSBM and a uniform order p = 6 in SDSBM for k = 2, with Figure 1 and 2 showing the Regions of Absolute Stability (RAS) are A-stable methods. Comparison of absolute error was made using the new methods of order 6 and 7, and the new block methods are more favorable than others, such as Sahi et al (2012), Nwachukwu G.C., and Mokwunyei N.E. (2018), and Sunday (2022), are presented in Tables 1 to 5, the solution curves are displayed in Figures 3 to 7. The derivations of the methods was done using the program codes (Maple), while the computation of the RAS, and results was done using the program codes (MATLAB).

5. Conclusion

A Modified Second Derivative Simpson's Block Methods (MSDSBM) with off-grid points is proposed and obtain discrete methods for solving Stiff system, it has been shown that collocation methods for solving ordinary differential equations can equally be derived through the LU Decomposition techniques approach, all the required additional equations are obtained from the same continuous formulation. In this study, a new block methods that is capable of solving higher-order initial value problems of ordinary differential equations is presented. The basic property of the block methods was investigated and found to be zero-stable, consistent, and convergent. The regions absolute stability of the block methods are all A-stable and of high order, and overcome the major constraints of stiff problems. The numerical results show that the methods are efficient and highly competitive with the existing methods cited in this paper.

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