ON THE NEGATIVE PELLIAN EQUATION

$x^2 = 8y^2 - 4$

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Abstract

The binary quadratic equation represented by the negative pellian $x^2 = 8y^2 - 4$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Keywords: Binary quadratic, Pell equation, Hyperbola, Parabola, Integral solutions

1.Introduction

Diophantine equation of the form $y^2 = Dx^2 - 1$, where D is a given positive square-free integer is known as pell equation and is one of the oldest Diophantine equation that has interesting mathematicians all over the world, since antiquity, J.L. Lagrange proved that the positive pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions whereas the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a centerium for the solvability of the pell equation $x^2 - Dy^2 = 1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solution whereas $y^2 = 54x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2-18]. More specifically, one may refer "The on-line encyclopedia of integer sequences" (A031396, A130226, A031398) for all values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not. In this communication, the negative pell equation given by is considered $x^2 = 8y^2 - 4$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

2. Method of Analysis:

The Diophantine equation under consideration is

$$x^2 = 8y^2 - 4 \tag{1}$$

The smallest positive integer solution (y_0, x_0) of (1) is

$$y_0 = 1,\ x_0 = 2$$

To obtain the other solutions of (1), consider the pellian equation

$$x^2 = 8y^2 + 1 \tag{2}$$

whose general solution $(\tilde{y}_n, \tilde{x}_n)$ is given by

$$\tilde{x}_n = \frac{1}{2}\left[\left(3+\sqrt{8}\right)^{n+1} + \left(3-\sqrt{8}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{y}_n = \frac{1}{2\sqrt{8}}\left[\left(3+\sqrt{8}\right)^{n+1} - \left(3-\sqrt{8}\right)^{n+1}\right] = \frac{1}{2\sqrt{8}} g_n$$

Applying Brahmagupta lemma between the solutions (y_0, x_0) and $(\tilde{y}_n, \tilde{x}_n)$, the general solution (y_{n+1}, x_{n+1}) of (1) is found to be

$$y_{n+1} = \frac{1}{\sqrt{8}} g_n + \frac{1}{2} f_n \tag{3}$$

$$x_{n+1} = f_n + \frac{\sqrt{8}}{2} g_n \tag{4}$$

Thus, (9) and (10) represent the integer solutions of the hyperbola (1).

A few numerical examples are given in the following Table 1:

Table 1: Examples

n	x_{n+1}	y_{n+1}
-1	2	1
0	14	5
1	82	29
2	478	169

The recurrence relation satisfied by the values of x_{n+1} and y_{n+1} are respectively

$$x_{n+3} - 6x_{n+2} + x_{n+1} = 0, n = -1,0,1.....$$

$$y_{n+3} - 6y_{n+2} + y_{n+1} = 0, n = -1,0,1.....$$

- *A few interesting relations among the solutions are given below:*

  - $3x_{n+1} - x_{n+2} + 8y_{n+1} = 0$
  - $x_{n+1} - 3x_{n+2} + 8y_{n+2} = 0$
  - $3x_{n+1} - 17x_{n+2} + 8y_{n+3} = 0$
  - $17x_{n+1} - x_{n+3} + 48y_{n+1} = 0$
  - $3x_{n+1} - 3x_{n+3} + 48y_{n+2} = 0$
  - $x_{n+1} - 17x_{n+3} + 48y_{n+3} = 0$
  - $y_{n+2} - x_{n+1} - 3y_{n+1} = 0$
  - $y_{n+3} - 6x_{n+1} - 17y_{n+1} = 0$
  - $3y_{n+3} - x_{n+1} - 17y_{n+2} = 0$
  - $x_{n+1} + 17y_{n+2} - 3y_{n+3} = 0$
  - $17x_{n+2} - 3x_{n+3} + 8y_{n+1} = 0$
  - $3x_{n+2} - x_{n+3} + 8y_{n+2} = 0$
  - $x_{n+2} - 3x_{n+3} + 8y_{n+3} = 0$
  - $3y_{n+2} - x_{n+2} - y_{n+1} = 0$
  - $3y_{n+3} - 6x_{n+2} - 3y_{n+1} = 0$
  - $x_{n+2} + y_{n+1} - 3y_{n+2} = 0$
  - $y_{n+3} - x_{n+2} - 3y_{n+2} = 0$
  - $x_{n+2} + 3y_{n+2} - y_{n+3} = 0$
  - $17y_{n+2} - x_{n+3} - 3y_{n+1} = 0$
  - $17y_{n+3} - 6x_{n+3} - y_{n+1} = 0$
  - $3x_{n+2} - x_{n+3} + 8y_{n+2} = 0$
  - $3y_{n+3} - x_{n+3} - y_{n+2} = 0$
  - $x_{n+3} + y_{n+2} - 3y_{n+3} = 0$

- *Each of the following expressions represents a cubical integer*

  - $\frac{1}{4}\left[\left(2x_{3n+4} - 10x_{3n+3}\right) + 3\left(2x_{n+2} - 10x_{n+1}\right)\right]$

- ❖ $\frac{1}{24}\left[\left(2x_{3n+5}-58x_{3n+3}\right)+3\left(2x_{n+3}-58x_{n+1}\right)\right]$
- ❖ $\frac{1}{4}\left[\left(16y_{3n+3}-4x_{3n+3}\right)+3\left(16y_{n+1}-4x_{n+1}\right)\right]$
- ❖ $\frac{1}{12}\left[\left(16y_{3n+4}-28x_{3n+3}\right)+3\left(16y_{n+2}-28x_{n+1}\right)\right]$
- ❖ $\frac{1}{68}\left[\left(16y_{3n+5}-164x_{3n+3}\right)+3\left(16y_{n+3}-164x_{n+1}\right)\right]$
- ❖ $\frac{1}{4}\left[\left(10x_{3n+5}-58x_{3n+4}\right)+3\left(10x_{n+3}-58x_{n+2}\right)\right]$
- ❖ $\frac{1}{4}\left[\left(80y_{3n+4}-28x_{3n+4}\right)+3\left(80y_{n+2}-28x_{n+2}\right)\right]$
- ❖ $\frac{1}{12}\left[\left(80y_{3n+5}-164x_{3n+4}\right)+3\left(80y_{n+3}-164x_{n+12}\right)\right]$
- ❖ $\frac{1}{68}\left[\left(464y_{3n+3}-4x_{3n+5}\right)+3\left(464y_{n+1}-4x_{n+3}\right)\right]$
- ❖ $\frac{1}{12}\left[\left(464y_{3n+4}-28x_{3n+5}\right)+3\left(464y_{n+2}-28x_{n+3}\right)\right]$
- ❖ $\frac{1}{4}\left[\left(464y_{3n+5}-164x_{3n+5}\right)+3\left(464y_{n+3}-164x_{n+3}\right)\right]$
- ❖ $\frac{1}{4}\left[\left(28y_{3n+3}-4y_{3n+4}\right)+3\left(28y_{n+1}-4y_{n+2}\right)\right]$
- ❖ $\frac{1}{24}\left[\left(164y_{3n+3}-4y_{3n+5}\right)+3\left(164y_{n+1}-4y_{n+3}\right)\right]$
- ❖ $\frac{1}{4}\left[\left(164y_{3n+4}-28y_{3n+5}\right)+3\left(164y_{n+2}-28y_{n+3}\right)\right]$

➢ *Each of the following expressions represents Biquadratic integer:*

- ❖ $\frac{1}{(4)^2}\left[\left(8x_{4n+5}-40x_{4n+4}\right)+4\left(2x_{n+5}-10x_{n+1}\right)^2-32\right]$
- ❖ $\frac{1}{(4)^2}\left[\left(64y_{4n+4}-16x_{4n+4}\right)+4\left(16y_{n+1}-4x_{n+1}\right)^2-32\right]$
- ❖ $\frac{1}{(12)^2}\left[\left(192y_{4n+5}-336x_{4n+4}\right)+4\left(16y_{n+2}-28x_{n+1}\right)^2-288\right]$
- ❖ $\frac{1}{(4)^2}\left[\left(40x_{4n+6}-232x_{4n+5}\right)+4\left(10x_{n+3}-58x_{n+2}\right)^2-32\right]$

- $\frac{1}{(12)^2}\left[(960y_{4n+4}-48x_{4n+5})+4(80y_{n+1}-4x_{n+2})^2-288\right]$
- $\frac{1}{(4)^2}\left[(320y_{4n+5}-112x_{4n+5})+4(80y_{n+2}-28x_{n+2})^2-32\right]$
- $\frac{1}{(12)^2}\left[(960y_{4n+6}-1968x_{4n+5})+4(80y_{n+3}-164x_{n+2})^2-288\right]$
- $\frac{1}{(68)^2}\left[(31552y_{4n+4}-272x_{4n+6})+4(464y_{n+1}-4x_{n+3})^2-9248\right]$
- $\frac{1}{(12)^2}\left[(5568y_{4n+5}-336x_{4n+6})+4(464y_{n+2}-28x_{n+3})^2-288\right]$
- $\frac{1}{(4)^2}\left[(1856y_{4n+6}-656x_{4n+6})+4(464y_{n+3}-164x_{n+3})^2-32\right]$
- $\frac{1}{(4)^2}\left[(112y_{4n+4}-16y_{4n+5})+4(28y_{n+1}-4y_{n+2})^2-32\right]$
- $\frac{1}{(24)^2}\left[(3936y_{4n+4}-96y_{4n+6})+4(164y_{n+1}-4y_{n+3})^2-1152\right]$
- $\frac{1}{(4)^2}\left[(656y_{4n+5}-112y_{4n+6})+4(164y_{n+2}-28y_{n+3})^2-32\right]$

- *Each of the following expressions represents a Nasty number:*

- $\frac{1}{4}\left[(96y_{2n+2}-24x_{2n+2})+48\right]$
- $\frac{1}{12}\left[(96y_{2n+3}-168x_{2n+2})+144\right]$
- $\frac{1}{68}\left[(96y_{2n+4}-984x_{2n+2})+816\right]$
- $\frac{1}{4}\left[(60x_{2n+4}-348x_{2n+3})+48\right]$
- $\frac{1}{12}\left[(480y_{2n+2}-24x_{2n+3})+144\right]$
- $\frac{1}{4}\left[(480y_{2n+3}-168x_{2n+3})+48\right]$
- $\frac{1}{12}\left[(480y_{2n+4}-984x_{2n+3})+144\right]$
- $\frac{1}{68}\left[(2784y_{2n+2}-24x_{2n+4})+816\right]$

- $\frac{1}{4}[(2784y_{2n+4} - 984x_{2n+4}) + 48]$
- $\frac{1}{4}[(168y_{2n+2} - 24y_{2n+3}) + 48]$
- $\frac{1}{4}[(984y_{2n+3} - 168y_{2n+4}) + 48]$

3. Remarkable Observations:

3.1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in Table 2: below

Table 2: Hyperbolas

S.No	Hyperbolas	(X_n, Y_n)
1	$8X_n^2 - Y_n^2 = 512$	$(2x_{n+2} - 10x_{n+1}, 28x_{n+1} - 4x_{n+2})$
2	$8X_n^2 - Y_n^2 = 18432$	$(2x_{n+3} - 58x_{n+1}, 164x_{n+1} - 4x_{n+3})$
3	$8X_n^2 - Y_n^2 = 512$	$(16y_{n+1} - 4x_{n+1}, 16x_{n+1} - 32y_{n+1})$
4	$8X_n^2 - Y_n^2 = 4608$	$(16y_{n+2} - 28x_{n+1}, 80x_{n+1} - 32y_{n+2})$
5	$8X_n^2 - Y_n^2 = 147968$	$(16y_{n+3} - 164x_{n+1}, 464x_{n+1} - 32y_{n+3})$
6	$8X_n^2 - Y_n^2 = 512$	$(10x_{n+3} - 58x_{n+2}, 164x_{n+2} - 28x_{n+3})$
7	$8X_n^2 - Y_n^2 = 512$	$(80y_{n+2} - 28x_{n+2}, 80x_{n+2} - 224y_{n+2})$
8	$8X_n^2 - Y_n^2 = 4608$	$(80y_{n+3} - 164x_{n+2}, 464x_{n+2} - 224y_{n+3})$
9	$8X_n^2 - Y_n^2 = 147968$	$(464y_{n+1} - 4x_{n+3}, 16x_{n+3} - 1312y_{n+1})$
10	$8X_n^2 - Y_n^2 = 4608$	$(464y_{n+2} - 28x_{n+3}, 80x_{n+3} - 1312y_{n+2})$
11	$8X_n^2 - Y_n^2 = 512$	$(464y_{n+3} - 164x_{n+3}, 464x_{n+3} - 1312y_{n+3})$
12	$8X_n^2 - Y_n^2 = 512$	$(28y_{n+1} - 4y_{n+2}, 16y_{n+2} - 80y_{n+1})$
13	$8X_n^2 - Y_n^2 = 18432$	$(164y_{n+1} - 4y_{n+3}, 16y_{n+3} - 464y_{n+1})$
14	$8X_n^2 - Y_n^2 = 512$	$(164y_{n+2} - 28y_{n+3}, 16y_{n+3} - 464y_{n+1})$

3.2. Employing linear combination among the solutions for other choices of parabola which are presented in Table 3: below

Table 3: Parabolas

SL.NO	Parabolas	(X_n, Y_n)
	$32X_n - Y_n^2 = 512$	$(8 + 2x_{2n+3} - 10x_{2n+2}, 28x_{n+1} - 4x_{n+2})$
2	$192X_n - Y_n^2 = 18432$	$(48 + 2x_{2n+4} - 58x_{2n+2}, 164x_{n+1} - 4x_{n+3})$
3	$32X_n - Y_n^2 = 512$	$(8 - 4x_{2n+2} + 16y_{2n+2}, 16x_{n+1} - 32y_{n+1})$
4	$96X_n - Y_n^2 = 4608$	$(24 - 28x_{2n+2} + 16y_{2n+3}, 80x_{n+1} - 32y_{n+2})$
5	$544X_n - Y_n^2 = 147968$	$(136 - 164x_{2n+2} + 16y_{2n+4}, 464x_{n+1} - 32y_{n+3})$
6	$32X_n - Y_n^2 = 512$	$(8 - 58x_{2n+3} + 10x_{2n+4}, 164x_{n+2} - 28x_{n+3})$
7	$32X_n - Y_n^2 = 512$	$(8 - 28x_{2n+3} + 80y_{2n+3}, 80x_{n+2} - 224y_{n+2})$
8	$96X_n - Y_n^2 = 4608$	$(24 - 164x_{2n+3} + 80y_{2n+4}, 464x_{n+2} - 224y_{n+3})$
9	$544X_n - Y_n^2 = 147968$	$(136 - 4x_{2n+4} + 464y_{2n+2}, 16x_{n+3} - 1312y_{n+1})$
10	$96X_n - Y_n^2 = 4608$	$(24 - 28x_{2n+4} + 464y_{2n+3}, 80x_{n+3} - 1312y_{n+2})$
11	$32X_n - Y_n^2 = 512$	$(8 - 164x_{2n+4} + 464y_{2n+4}, 464x_{n+3} - 1312y_{n+3})$
12	$32X_n - Y_n^2 = 512$	$(8 - 4_{2n+3} + 28y_{2n+2}, 16y_{n+2} - 80y_{n+1})$
13	$192X_n - Y_n^2 = 18432$	$(48 - 4y_{2n+4} + 164y_{2n+2}, 16y_{n+3} - 464y_{n+1})$
14	$32X_n - Y_n^2 = 512$	$(8 - 28y_{2n+4} + 164y_{2n+3}, 16y_{n+3} - 464y_{n+1})$

3.3. *Consider* $p = x + y$, $q = y$. *Observe that* $p > q > 0$. *Treat* p, q *as the generators of the Pythagorean triangle* $T(\alpha, \beta, \gamma)$, *where*

$$\alpha = 2pq,\ \beta = p^2 - q^2,\ \gamma = p^2 + q^2$$

Then the following interesting relations are observed:

a) $\alpha - 4\beta + 3\gamma = 4$

b) $5\alpha - \gamma = 16\frac{A}{P} + 4$

c) $\frac{2A}{P} = xy$

d) $3\alpha - 2\beta + \gamma - \frac{8A}{P} = 4$

4. Conclusion

In this paper, we have presented infinitely many integer solutions for all hyperbola represented by the negative pell equations $x^2 = 8y^2 - 4$. As the binary quadratic Diophantine equation are rich in variety, one may choices of negative pell equations and determine their integer solutions along with suitable properties.

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