Orbits of Permutation Groups

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Abstract

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If no element of G moves any subset of Ω by more than m points (that is, if $|\Gamma^g \setminus \Gamma|\leq m$ for every $\Gamma subseteq Omega$ and $g\in G$), and the lengths of all Orbits are not equal to 2.
Then the number t of G-orbits in Ω is at most $2m-2$.
Moreover, the groups attaining the maximum bound $t=2m-2$ will be classified. \vspace{.4cm}

Keywords : **permutation group; bounded movement; orbits**
2000 AMS classification subjects: 20B25

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Introduction:

small\hspace{.5cm} Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a Positive integer. If for a subset Γ of Ω the size $|\Gamma^g \setminus \Gamma|$ is bounded, for $g\in G$, we Define the movement of Γ as move (Γ) = max$_{g \in G} |\Gamma^g \setminus \Gamma|$. If move ($\Gamma)\leq m$ for all $\Gamma \subseteq \Omega$,then G is said to have {\it bounded Movement} and the {\it movement} of G is define as the maximum Of move(Γ) over all subsets Γ, that is,
$$m:=move(G):= sup\{ |\Gamma^g\setminus\Gamma||\Gamma\subseteq\Omega, g\in G\}.$$ This Notion was introduced in [3]. By [3, Theorem 1], if G has bounded Movement m, then Ω is finite. Moreover both the number of G-orbits in Ω and the length of each G-orbit are Bounded above by linear functions of m. In particular it was Shown that the number of G-orbits is at most $2m-1$. In this Paper we will improve this bound to $2m-2$, if the lengths of All orbits are not equal to 2. If $m=1$, then $t=1$, $|\Omega|=2$ or 3 and G is Z_{2} or Z_{3} or S_{3}. So in this paper

We suppose that m is greater than 1. In this paper we obtain the maximum bound of $2m-2$ for the number of G-orbits and give a Classification of all groups for which the bound $2m-2$ is Attained. We shall say that an orbit of permutation group is Nontrivial if its length is greater than 1. We use the notation $K:P$ for semi-direct product of K by P with normal subgroup K. The main result is the following theorem.
\noindent{\bf Theorem 1.1.} Let m be a positive integer and Suppose that G is a permutation group on a set $\Omega $ such that G has no fixed points in Ω, and G has bounded Movement equal to m . If the length of all orbits are not equal to 2, then the number t of G-orbits in Ω is at most $2m-2$. Also
if $t=2m-2$, then $m_{1}= m-1$ is a power of 2, and G is of order 32^{m-1}, all G-orbits have length 2, except one of them has length 3, and the point wise
Stabilizers of the G-orbits are precisely the $2m-3$ Distinct subgroups of
G of index 2 and one subgroup of index 3.\\
%\end{theorem}
\indent Note that an orbit of a permutation group is non trivial if its length is greater than 1. The groups described below are Examples of permutation groups with bounded movement equal to m Which have exactly $ 2m-2 $ nontrivial orbits.\\
\section{Examples and Preliminaries} Let
$1\neq g \in G$ and suppose that g in its disjoint cycle\\
Representations has t nontrivial cycles of lengths
$l_{1},...,l_{t}, say$. We might represent g as \\$g =
(a_{1}a_{2}...a _{l_{1}})(b_{1}b_{2}...b
_{l_{_2}})...(z_{1}z_{2}...z _{l_{t}})$. Let $ \Gamma(g)$ denote a subset of Ω consisting $ \lfloor l _{i}/2 \rfloor $ points from the ith cycle , for each i, chosen in such a way that $\Gamma(g)^g \bigcap\Gamma(g)$ = \O. For example ,we could choose \\$\Gamma(g) =
\{a_{2},a_{4},...,a_{k_{1}},b_{2},b_{4},...,b_{k_{2}},...,z_{2},z_{4},...,z_{k_{t}}\},$
where $k_{i} = l_{i}-1 $ if $ l_{i} $ is odd and $ k_{i} = l_{i} $ if $ l_{i} $ is even. Note that $\Gamma(g)$ is not uniquency determined as it depends on the way each cycle is written . For any set $\Gamma(g)$ consists of every point of very cycle of g. From the definition of $\Gamma(g)$ we see that$$|\Gamma(g)^g\setminus \Gamma(g)| = |\Gamma(g)| =
\sum_{i=1}^{t} \lfloor l_{i}/2 \rfloor.$$ The next lemma shows that this quantity is an upper bound for
$|\Gamma^g\setminus\Gamma|$ for an arbitrary subset Γ of Ω. \\
\noindent{\bf Lemma 2.1.} [5, Lemma 2.1] Let G be a permutation group on a set Ω and suppose that
$\Gamma\subseteq\Omega$. Then for each $g\in G$,
$|\Gamma^g\setminus\Gamma|\leq\ \sum_{i=1}^{t}\lfloor l_{i}/2
\rfloor $, where l_{i} is the length of the ith cycle of g

and t is the number of nontrivial cycles of g in its disjoint cycle representation. This upper bound is attained for $\Gamma=\Gamma(g)$ defined above. \\ Now in the following examples we will show that there
are families of groups having bounded movement equal to $m = 1+2^{d-1}$ and exactly $t=2m-2$ nontrivial orbits.\\
%\end{Lemma}. \\
\noindent{\bf Example 2.2.} For a positive integer d and a prime number 3, let $G_{1}:=\langle(123)\rangle\cong Z_{3}$ be a permutation group on $\Omega_{1}:=\{1,2,3\}$. Moreover, suppose that $G_{2}:= Z_{2}^{d}$, and $H_{1},...,H_{t}$ be all subgroups of index 2 in G on $\Omega_{2}:= \bigcup_{i=1}^{{2^{d}-1}}\Omega_{2i}$, where Ω_{2i} denotes the set of two cosets of H_{i} in G_{2}, $1 \leq i \leq t=2^{d}-1$. Then G_{2} has movement equal 2^{d-1} and also $(2^{d}-1)$ nontrivial orbits in Ω_{2}. Now we consider the direct product $G:=G_{2} \times G_{2}$ as a permutation group on Ω which is the disjoint union of Ω_{1} and Ω_{2}, and G_{1} and G_{2} act trivially on Ω_{2} and Ω_{1}, respectively. Then G has movement $1+2^{d-1}$ and $2m-2$ nontrivial orbits in Ω. The set Ω splits into $2^{d}=2m-2$
orbits under G, which are Ω_{1} and also 2^{d-1} orbits of length 2 in Ω_{2} . In particular, none of them is trivial.\\
\noindent{\bf Example 2.3.} Let d, G_{2 and Ω_{2} be as in Example 2.2. Suppose that the permutation group $G_{1}:=Z_{3}:Z_{2}$ on Ω_{1} of length 3 is the symmetric group S_{3}. Then $ G:= G_{1} \times G_{2}$ is a permutation group on $\Omega:= \Omega_{1}\cup\Omega_{2}$ (as in Example 2.2) with bounded movement $m = 1+2^{d-1}$ and $2m - 2 $, non trivial orbits in Ω.\\
\noindent{\bf Example 2.4.} Let d, G_{1, G_{2} and Ω_{2} have the same meaning as Examples 2.2 and 2.3
. Suppose that the
permutation group $G_{1}:=Z_{3}:Z_{2}$ on Ω_{1} of length 3 is a Ferobenius group with complement $Z_{2}=<u>$ and kernel Z_{3} of order
32^{d} for some positive integer d. Then \\ $G_{1}\times G_{2}=(Z_{3}:\langle u\rangle)\times G_{2}$\\ = $Z_{3}:(\langle u\rangle \times G_{2})$ where G_{2} acts on Z_{3} trivially\\ = $Z_{3}:(\langle u\rangle \times \langle g\rangle Z_{2}^{d})$ where $G_{2}= \langle g\rangle Z_{2}^{d}$ for $g\in G_{2}$.\\ We them have a subgroup $Z_{3}:(\langle xg\rangle Z_{2}^{d})$ of $G_{1} \times G_{2}$, which is a permutation group meeting the bound. As we will see in the proof of Theorem 2.3, these groups are isomorphic to $(Z_{3}:Z_{2})\times Z_{2}^{d}$. \\
When $m>1$, the classification
in Theorem 1.1 follows immediately from the
following theorem about subsets with movement m.\\
\\\noindent{\bf Theorem 2.3.} Let $G\l Sym(\Omega)$ be a

permutation group on a set Ω with $ t $ orbits for positive integer t,
Such that the length of all orbits are not equal
to 2. Moreover suppose that $\G\s\Omega$ such
that move $(\G) = m
>1$. Then
$t\leq 2m-2$ and the equality holds iff\\
(1) m is the sum of 1 and a power of 2;\\
(2) All G-orbits of G have lengths 2 except one orbit, say Δ, of length 3;\\
(3) The permutation group G_{1} induced by G on Ω_{1} is Z_{3} or a Frobenius group $Z_{3}:Z_{2}$;\\
(4) The permutation group G_{2} induced by G on $\Delta^{'}$ is elementary abelian of order $2^{d}=2m-2$, and the pointwise stabilizers of the G_{2}-orbits are precisely the $2^{d}-1$ disjoint subgroups of G_{2} of index 2;\\
(5) G is isomorphic to either $Z_{3}\times Z_{2}^{d}$, $(Z_{3}: Z_{2})\times Z_{2}^{d}$, or $(Z_{3}: Z_{2})\times Z_{2}^{d-1}$, where $2^{d}=2m-2$;\\
(6) If one of the G-orbits is 3, then the t different G-orbits are (isomorphic to) the coset spaces of the $2^{d}=2m-2$
Different subgroups of index 2 in G.

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Section Proof of Theorem 2.3.

From the characterizations of groups having bounded movement equal to m, and having $2m-1$ orbits (see [6]), we see that an permutation group can have at most $2m-2$ nontrivial orbits (see [10], Theorem 1). Indeed G can have $2m-2$ nontrivial orbits as we see Examples 2.2 and 2.3. \\ Let $\Omega_{1}, \Omega_{2}, ..., \Omega_{d}$, be d orbits of G, $\Delta=\bigcup_{i=1}^{d}\Omega_{i}$, which $d<t$, $\Delta^{'}=\Omega\setminus\Delta$ and K the pointwise stabilizer of $\acute{\Delta}$. Then $K \lhd G$. For $g\in G$ we denote by $fix(g)=\{\alpha\in \Omega| g(\alpha)=\alpha\}$ and $Supp(g)= \{\alpha\in \Omega| g(\alpha)\neq\alpha\}$ the set of fixed points of G and the support of g, respectively. By referring to the results in $[6]$ for the case having the maximum bound of orbits, we have the following facts: \\(i) $m-1$, is a power of 2.\\(ii) The permutation group induced by G/K on $\Delta^{'}$ is an elementary abelian 2-group Z_{2}^{d} of order $2^{d}=2m-2$.\\(iii) The permutation group induced by G/K on $\Delta^{'}$ has $2m-3$ nontrivial orbits and each orbit has lenght 2.\\(iv) Each nontrivial element of G/K permutes exactly $m-1$ of the $2m-3$ orbits. \\ (1) and (4) follow from (i) and (ii), respectively. By (iii) G has only one orbit which is not of length 2, say Δ with $|\Delta|=n_{1}$. Since every p-element of G is a p-cycle and is contained in K, K is transitive on Δ. Note that, $|K|=\sum_{k\in K}|fix(k)|$ and $|G|=\sum_{g\in G}|fix(g)|$. By (iv) for each $g\in G\setminus K$,\\
$m\geq |\Gamma(g)|=|\Gamma_{\Delta}(g)|+|\Gamma_\Delta^{'}(g)|=

|\Gamma_{\Delta}(g)|+m-1$. \\Hence $|\Gamma_{\Delta}(g)|\leq 1$ and so $|Supp(g)\cap \Delta|\leq 2$. Thus $|Fix(g)\cap \Delta|\geq n_{1}-2$. Thus \\$|G|=\sum_{g\in G}|fix(g)|=\sum_{k\in K}|fix(k)|+ \sum_{g\in G\setminus K}|fix(g)|\geq |K|+|G\setminus K|(n_{1}-2)$,\\which means $n_{1}=3$. This gives (2).

We now prove (3) and (5). If G_{1} is regular, then G_{1} and K are Z_{3}. Thus $G\simeq Z_{3}\times Z_{2}^{d}$, where Z_{3} and Z_{2}^{d} acts trivially on $\Delta^{'}$ and Δ, respectively. Suppose G_{1} is not regular. Since G_{1} is of order 3.2^{d-1}, it is soluble. Moreover it is a Frobenius group (see Theorem 11.6 in [11]). Thus $G_{1}=Z_{3}: C$ where Frobenius complement C is a subgroup of $Aut(Z_{3})\cong Z_{2}$. Thus $G_{1}=Z_{3}: Z_{2}$. we let $Z_{2}=<u>$ and $Z_{3}=<v>$, and write $G=\{v^{i}u^{j}s| s\in Z_{2}^{d}\}$. Note that v lies in G. If u lies in G, then $G=(Z_{3}: Z_{2})\times Z_{2}^{d}$. If $u\not\in G$, u^{2} lies in G. We then consider a subgroup $P=\{s\in Z_{2}^{d}| s\in G\}$ and a subset $Q=\{s\in Z_{2}^{d}| us\in G\}$ of Z_{2}^{d}. Since the permutation group induced by G/K on $\Delta^{'}$ is an elementary abelian 2-group Z_{2}^{d}, we have $P\cap Q= \phi$ and $P\cup Q=Z_{2}^{d}$. If $s^{'}$ and $s^{''}$ lie in Q, then $us^{'}us^{''}\in G$ and so does $s^{'}s^{''}
\in G$. This means $Q\subset \alpha P$ for some $\alpha\in Z_{2}^{d}\setminus P$ and $Z_{2}^{d}=P\cup \alpha P$. Hence
\begin{center}
$G=\{v^{i}u^{2j+1}\alpha t|t\in P\}\cup\{v^{i}u^{2j}t|t\in P\}$\\=$\{v^{i}(u\alpha)^{j}t|t\in P\}$.\\
\end{center}
Let $C=\{v^{i}(u\alpha)^{j}\}$. Then $P\cap C=\{1\}$ and $CP=G$. Since P and C are normal subgroups of G, we have $G\simeq C\times P$. Since $C=\{v^{i}(u\alpha)^{j}\}\simeq Z_{3}: Z_{2}$ and $P\simeq Z_{2}^{d-1}$, we have $G\simeq (Z_{3}: Z_{2})\times Z_{2}^{d-1}$. Thus the proof of Theorem 2.3 is complete.
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