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Digital-Concrete Materials: Revisiting Fröbel in Sketchpad Tasks

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ABSTRACT

The present paper sets out to revisit Fröbel's play theory, through the open-ended instructional materials, designed for pupils' learning which he bequeathed to us. Many researchers have highlighted the advantages of digital or computer concrete materials including DGS manipulatives for teaching and learning. In terms of the present study, it is interesting to mention the introduction of Fröbel's first Gift that I adapted in the DGS environment, designed to provide a play-based way of presenting/inquiring about geometric objects. The proposed DGS materials can be displayed, inquired about, and managed through properly set-up tasks, using linking visual active representations. The dynamic notions (e.g., dynamic point, segment, instrumental decoding, hybrid-dynamic objects, etc.), are taken as given and form the specific theoretical basis for the required processes. Dynamic interdependencies of tools in various sequential steps will be considered for the idea of building DGS Gifts, linked to the pupils' level of conceptualization.

KEYWORDS

Digital-concretematerials; DGS environment; Fröbel's Gifts

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1. Introduction: Digital-Concrete Materials and Fröbel (/Froebel) Gifts

Manipulatives or concrete representations are objects which are designed to mediate between a particular mathematical concept and the way pupils learn the concept. Pupils can manipulate them by touching or moving, and thus are concrete means (Baroody, 1989; Dienes, 1960;Van de Walle et al., 2005). In their substantial study, Bartolini Bussi, DainaTaimina,& Masami Isoda (2009) report Comenius (1592-1670) and his Didactica Magna (1657) on knowledge construction in an extended excerpt, which I have include, as I think it applies to my study:

> Everything must be presented to the senses as much as possible; to wit, the visible to the eye, the audible to the ear, odors to the sense of smell, the tastable to the taste, and the touchable to the sense of touch; and, whenever something can be grasped by more than one sense at one time, let it be presented to them at one time. One may, however, if the things themselves cannot be presented, use representations of them, such as models and pictures. […] It is a mistake to let rules in an abstract form go before, and afterwards explain them in examples. For the light must go before him for whom it is intended to shine. […](Comenius, 1657, cited in Bartolini Bussi et al., 2009, p. 19)

In their article (Bartolini Bussi et al., ibid, p. 20) they also mention Friedrich Fröbel (1782–1852), a German educator, the founder of the Kindergarten Movement, who designed open-ended instructional materials, called the "Gifts", which could help pupils to progress "*from the material to the abstract*". As Ronge & Ronge (1858, Introduction, p. xiii) write "*Fröbel supplied the children with material, with which, as will be seen from the description of the various Gifts, the children can produce an unlimited number of forms*". In the figures below (Figure 1) Ronge & Ronge (1858) describe several movements of a soft ball that "*is the most convenient and the best adapted. In form it is round, like the heavenly bodies and in appearance it combines the idea of the finite with the infinite*" (p. 1). In pictures 1 and 2, the ball is suspended on a string and swung at the pupil to help them understand the difference between 'here' and 'there'. In pictures 8 and 9, the ball is turning to the left and right and the child thus observes the form of the curve. In pictures 17 and 18, the string is shortened, which causes the circles to become smaller. In pictures 19 and 24, the ball is allowed to bounce higher or lower on the table. In pictures 22 and 23, the string is getting always larger or smaller.

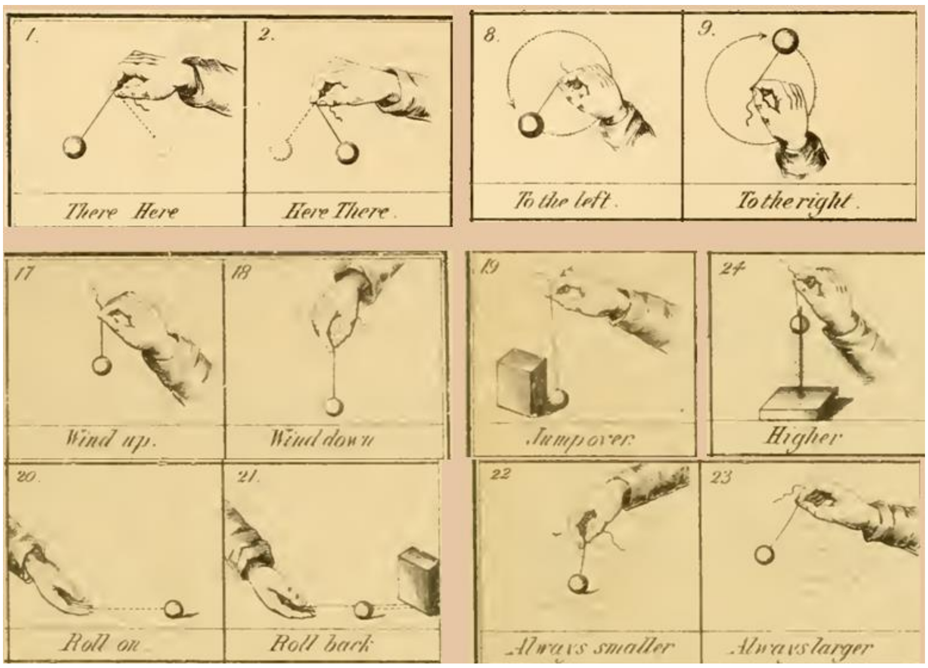


Figure 1. Selected pictures from the experiments with the first Gift (Joh & Bertha Ronge, 1858) (adapted)

Many researchers highlight the advantages of computer manipulatives including DGS manipulatives for teaching and learning (e.g., Clements &Mcmillen, 1996; Clements &Sarama, 2007; Patsiomitou, 2022 a, b). In terms of the present study, it is interesting to mention the introduction of Fröbel first Gift that I adapted in the DGS environment, designed to provide a mindful play-based way of discovering/constructing geometric objects (Papert, 1980).In this article, I shall also present a kind of DGS physical material (e.g., DGS leaves). Fröbel proposed the first Gift to be given to infants and very young learners. "*His intention was that, through holding, dropping, rolling, swinging, hiding, and revealing the balls, the child may acquire knowledge of objects and spatial relationships, movement, speed and time, colour and contrast, and weights and gravity*" (website [1]).

According to Fröbel "*Knowledge is there in the world, and the task is to see the geometrical, mathematical structures uniting everything […] Froebel's theory of play provides the child with a possibility to find knowledge of this world system.*" (Johansson, 2022, p.73). Fröbel (1885) focuses on the need for interaction between pupil /student, teacher and content of mathematical Knowledge (i.e., the "Didactic Triangle") (e.g., Steinbring, 2005). Olive et al. (2010) created an adaptation of the Didactic Triangle (Figure 2) in order to incorporate the use of Technology, which I also adapt for the current study. As they write "*We add "technology" as a fourth vertex of the didactic triangle, transforming it into a 3-D tetrahedron, creating three new triangular faces, each face illustrating possible inter-relationships among student, teacher, mathematical knowledge and tech*nology" (Olive et al., 2008, p.136).

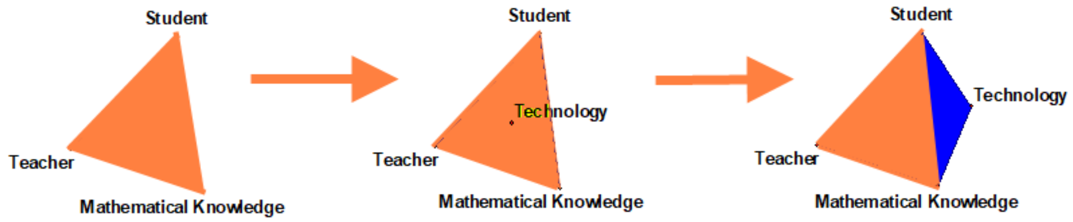


Figure 2.Transforming the Didactic Triangle into the Didactic Tetrahedron (Olive et al., 2010, p.136) (adapted)

Every Gift created in the DGS environment is underpinned by scientific principles that relate (a) to the mathematical concepts that underlie the digital representation and (b) to the *instrumental decoding* (Patsiomitou, 2011, 2012, 2021a,b) I used to present these concepts in the DGS environment. This is also the basis behind the invention and the creation of the new innovative digital-concrete material. The dynamic notions (e.g., dynamic point, segment, instrumental decoding, hybrid-dynamic objects etc.), are taken as given and form the specific /particular theoretical basis for the required processes (Patsiomitou, 2008a, b, 2010, 2011, 2012, 2018, 2019a, b, 2021a, b).The DGS tasks proposed here can also be used by children of students in Primary education. The experience -by using these materials- can become a high-level tool for the conceptions of meanings and the development of abstract concepts. It is what Balacheff (2010) in the following excerpt reports as *dynamic set of conceptions*.

> […] I can derive a definition of knowing as the characterization of a dynamic set of conceptions. This definition has the advantage of being in line with our usual use of the word "knowing" while providing grounds to understand the possible contradictions evidenced by learners' behaviours and their variable mathematical development. A conception is a situated knowing; in other words, it is the instantiation of a knowing in a specific situation detailed by the properties of the milieu and the constraints on the relations (action/feedback) between this milieu and the subject. (Balacheff 2010, p. 18)

How can we provide pupils with a "*journey of knowledge*" that is full of experiences through the educational design of the material, when this presupposes that the sequence in which the tools are used is properly designed? What kind of knowledge must be developed first during the teaching and learning of mathematics if pupils /students are to understand mathematical concepts? This argument recognizes and underlines the force of Kant's argument (1929/1965), that: "*there can be no doubt that all our knowledge begins with experience. […] knowledge consists in the determinate relation of given representations to an object*".

2. Experiential Learning

Piaget (1937/1971) considered that students' thinking becomes more sophisticated with biological maturity. Students build on their own intellectual structures as they grow up. Piaget introduced the development of student's thinking in stages, based on the process of equilibration. Von Glasersfeld (1995, p.68) describes equilibration as the process "*when a scheme, instead of producing the expected result, leads to a perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibriu*m". Consequently, *disequilibration*(Piaget, 1937) situations force students to reorganize their cognitive structures, when a conceptual structure does not act in line with their expectations. The reorganization of the individual's schemata involves the subprocesses or the mechanisms of *accommodation or assimilation* (Piaget, ibid.) which correspond to modifying the pre-existing schemata and building new schemata in the student's mind or interpreting the new information according to pre-existing schemata. Many times, students face *misconceptions* (e.g., Nesher, 1987; Swedosh, & Clark, 1998) and *cognitive conflicts* (e.g., Moritz,1998; Watson & Moritz, 2001) (see also, Patsiomitou, 2019a, Chapter 4). According to Piagetian theory, a young learner (a child aged 4–5) is not able to learn what is presented to them mindfully and deliberately or thoughtfully, since they have not yet acquired the required mental standards and qualities. In a constructivist frame, cognitive conflict is a basic component in the learning process (Karmiloff-Smith &Inhelder, 1974) and very important for the development of students' mathematical thinking. If the student overcomes this contradiction s/he is able to mental growth. According to Piaget &Inhelder (1956), the maturity of the student's thinking plays an important role in helping them reach the mental transformations stage.

Experiential Learning or learning through experience is a theory developed by David A. Kolb (1984). According to Kolb & Kolb (2013) the theory, described in detail in Experiential Learning: *Experience as the Source of Learning and Development* (Kolb, 1984), is built on six propositions that are shared by these scholars:

- *Learning is best conceived as a process, not in terms of outcomes.[…]*
- *All learning is re-learning.[…]*
- *Learning requires the resolution of conflicts between dialectically opposed modes of adaptation to the world.[…]*
- *Learning is a holistic process of adaptation to the world. […]*
- *Learning results from synergetic transactions between the person and the environment. […]*
- *Learning is the process of creating knowledge. (p.6-7)*

According to Kolb & Kolb (2009) "Immediate or concrete experiences are the basis for observations and reflections. These reflections are assimilated and distilled into abstract concepts from which new implications for action can be drawn. These implications can be actively tested and serve as guides in creating new experiences" (p.8) (Figure 3).James Zull (2002, 2011) describes a connection between ELT and neuroscience research, depicting a brain in which the sub-processes of experiential learning are related to the processes of brain's functions (Figure 4). As Zull(2002) describes

> "Put into words, the figure illustrates that concrete experiences come through the sensory cortex, reflective observation involves the integrative cortex at the back, creating new abstract concepts occurs in the frontal integrative cortex, and active testing involves the motor brain. In other words, the learning cycle arises from the structure of the brain." (Zull, 2002, pp. 18-19; 2011 cited in Kolb & Kolb, 2013)

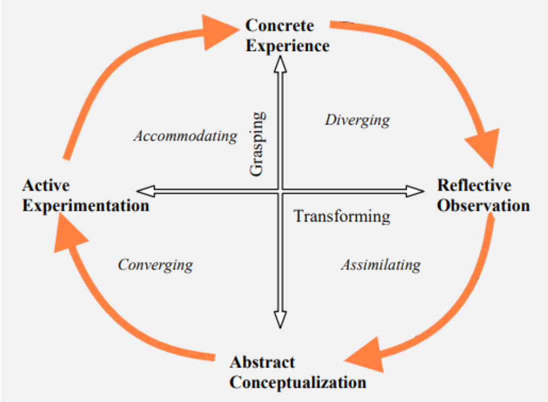
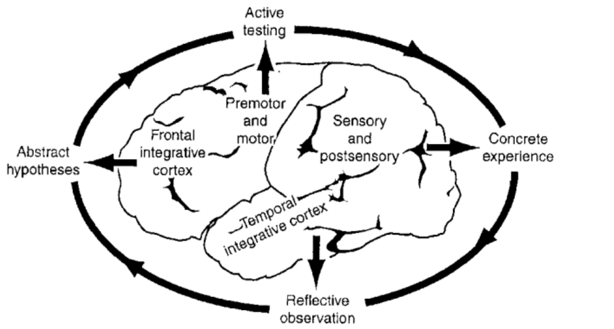
	

Figure 3. The Experiential Learning Cycle (Kolb, 1984, cited in Kolb & Kolb, 2009, p.8)	***Figure 4.*** The Experiential Learning Cycle and Regions of the Cerebral Cortex. (Zull 2002, pp. 18-19; 2011, cited in Kolb & Kolb, 2013, p.9)

Kolb & Kolb (2013) depict an amazing idea in a spiral that illustrates the spiraling learning process. As they state the spiraling learning "*begins with activity, moves through reflection, then to generalizing and abstracting and finally to transfer*" (Henton, 1996, p. 39, cited in website [2]).

3. Diagrams, DGS diagrams and Duval's cognitive apprehension

A crucial issue concerning geometrical meanings relates to the nature of the geometric reasoning students develop through problem solving tasks. The visual images that are used in geometry to represent mathematical objects, both those we perceive in space and those that exist only in our minds have a dual nature: *concrete object* on the other hand they are also *a general* devoid of empirical constraints. According to Mesquita (1998, pp.185-186) "representing a concept or a situation in geometry, the material trace can suggest two different possibilities: (a) '*finiteness*' in the sense of finite and diversified forms (*Gestalten*) in its spatio-temporality (b) geometrical *Form* in its '*ideal objectiveness*,' detached from the material constraints linked to external representation."

Chen & Herbst (2005) investigate the duality of diagrams "*being at the same time representations of concepts and actual objects*". Diezmann (2005, p.281) also states that diagrams have three key cognitive advantages in problem solving: "[…] (a) Diagrams facilitate the conceptualization of the problem structure, which is a critical step towards a successful solution (van Essen &Hamaker, 1990). (b) Diagrams are an inference-making knowledge representation system (Lindsay, 1995) that has the capacity for knowledge generation (Karmiloff-Smith, 1990). (c) Diagrams support visual reasoning, which is complementary to, but differs from, linguistic reasoning (Barwise &Etchemendy, 1991)." Vergnaud (1998) in his study "*A Comprehensive Theory of Representation for Mathematics Education*" argues that "*representation is not a static thing but a dynamic process that borrows a lot from the way action is organized. […].*" (p. 167). Vergnaud (1998) "*opposes the metaphor of the Aristotelian triangle because the metaphor of triangle is too static and does not offer any insight for the representation of relationships*" (Mainali, 2021) and states (Vergnaud, 2009, p.93):

> "Representation is a dynamic activity, not an epiphenomenon that would accompany activity without feeding it or driving it. […] it organizes and regulates action and perception; at the same time, it is also the product of action and perception. Therefore, the operational form of knowledge must be considered as a component of representation. […]".

The dynamic diagrams/or dynamic representations constructed in a DGS environment result from sequences of primitives expressed in geometrical terms chosen by the user.

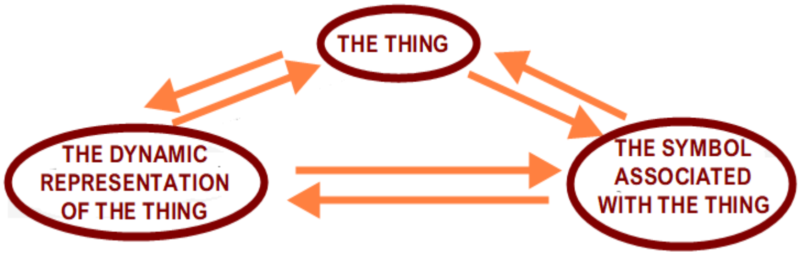


Figure 5. An adaptation to the metaphor of the (Aristotelian) triangle (see also, Vergnaud, 1998, p. 168, cited in Patsiomitou, 2019a, Chapter 2)

Dynamic diagrams have two important characteristics: (a) they are new mathematical objects that map a temporal past of specification and definition onto a present graphical configuration and also onto a future potential for manipulation and possible constrained response. Shape becomes infinitely plastic within the software (Jackiw, 2006) (b) they acquire quasi-independence of the user once they have been created: when the user drags one element of the diagram, it is modified according to the geometry of its constructions rather than the wishes of the user (Laborde, 2005).

Duval (1995, p.145-147) provides an analytic framework for analysing the semiotics of geometric objects as theoretical and abstract objects. Duval identifies or distinguishes four types of cognitive apprehension, namely how we perceive (with our sensory system) and conceive (in our mind) a figure or a diagram. These types of cognitive apprehension are the following (see also Deliyianni et al., 2009; Forsythe, 2014; Jones, 1998; Patsiomitou, 2011, 2012, 2018, 2019a, b):

perceptual apprehension: this is what is recognised at first glance; how one perceives a figure, what are the sub-figures in the figure; in other words what one can view in the figure or perceive in regard of the objects that belong to the figure.

sequential apprehension: how one understands the order of the construction steps; what are the geometric properties and definitions used for the construction of the figure. Using a DGS or computing environment generally a student can enrich his understanding of the different paths that can be used for the same construction of a figure (see also Gomes and Vergnaud, 2004, cited in Forsythe, 2014, p.40)

discursive apprehension: how one verbalizes the construction steps and explicate/interpret the construction steps using reasoning; "the definition of a geometrical object and a description of its construction are part of discursive apprehension" (Forsythe, 2014, p.40)

operative apprehension, how one operates the figure "which involves manipulating the figure mentally or physically to provide an insight into a problem" (Jones, 1998, p. 31). "Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way" (Deliyianni et al. 2009, p. 697).

Duval (1999) in his study "*Representation, vision and visualization: cognitive functions in mathematical thinking. Basic issues for learning*", describes three kinds of operations delimited by how a given figure is transformed:

"*The mereologic way*: you can divide the whole given figure into parts of various shapes [...] and you can combine these parts in another whole figure or you can make appear new subfigures. [...] We call «*reconfiguration*» the most typical operation.

The optic way: you can make a shape larger or narrower, or slant, as if you would use lenses. In this way, without any change, the shapes can appear differently [...].

The place way: you can change its orientation in the picture plane. It is the weakest change. It affects mainly the recognition of right angles, which visually are made up of vertical and horizontal lines" (Duval, 1988, pp. 61-63; 1995, p.147).

The mereologic, the optic way and the place way constitute what Duval defined as "*the operative apprehension*" of the figure, which according to him differs from the perceptual apprehension "*because perception fixes at the first glance the vision of some shapes and this evidence makes them steady*" (p.19) [...] Operative apprehension is [also] independent of discursive apprehension" (p.21). According to Duval "*Visualization consists only of operative apprehension* [and]…*operative apprehension is independent of discursive apprehension*" (1999, p. 21).

4. The design of the first Fröbelian DGS-Gift

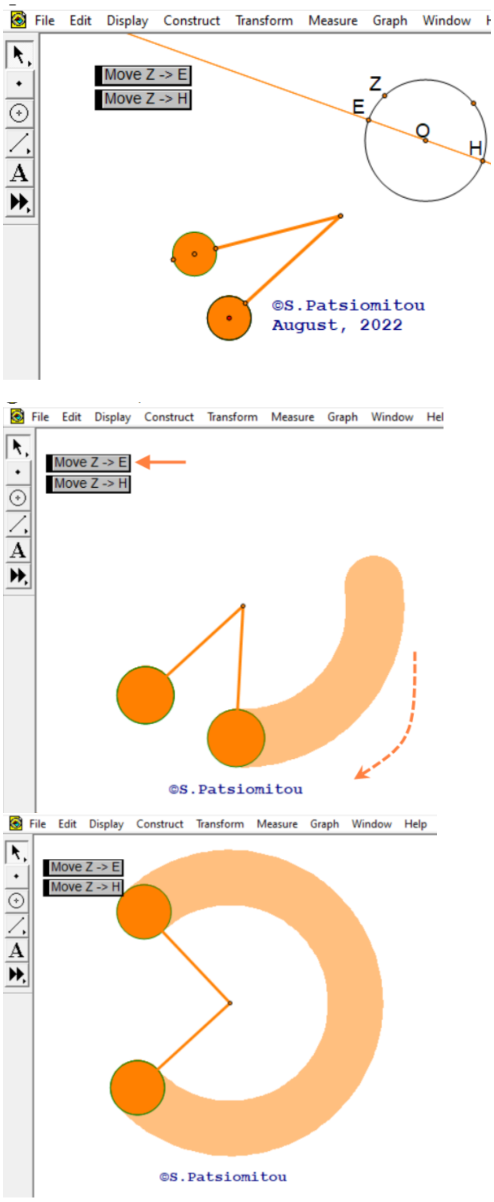
"A string is fastened to the ball; it is swung round; the child is taught to observe the two forces employed. The circular motion is interrupted, and the child perceives that it is still held to the centre. It is again swung and allowed to fly from the centre; the child is induced to observe the differences of the motions, and led to perceive the relation between the centrifugal and the centripetal force"(Ronge & Ronge, 1858, p.6)

In the diagram on the left, a "ball" is dynamically "tied" at a point and can be moved to right and left but also in a circular line. In the software, movement is achieved using two action buttons which move points on an auxiliary circle. This circle can be hidden for elementary school pupils, or not for junior high school students. The action will help students to visualize the way the points are moved as a covariation action.

We can propose to a student that they press the first button and observe the ball's clockwise movement. We can connect the interior of the ball to a trace, so the ball's movement is mapped onto the working area on the desktop. Pressing the next button enables counter clockwise movement. Students can observe the ball moving and the curved or circular arcs as they are formed simultaneously. Traces of this object are important for understanding basic geometric concepts such as a semicircle, a circle or a sector.

In the diagram on the left, we can stop the movement of the ball when the angle of the sector is equal to 270°. Our aim is for the students to visualize the different angles formed by the arcs of the ball's trajectory. The ultimate goal is to form a circle when the ball returns to its place.

If we choose to capture the arcs traced left by the ball as well the radius-segment, then a circular sector is formed; this is a very important geometric concept.



***Figures 6* a, b, c.** Moving the ball in a cyclical process using action buttons

Continuing the movement, a semicircle is captured as an extension of the concept of the circular sector and the circle. These movements help the students perceive the concept of area. If we choose to shorten the radius, then the students can observe a smaller circular sector. With the trace tool, we can connect the point of intersection to the diametrically opposite point of the circle. The movement creates different arcs--the semi-circular arc, as well as the circle (Figure 7).

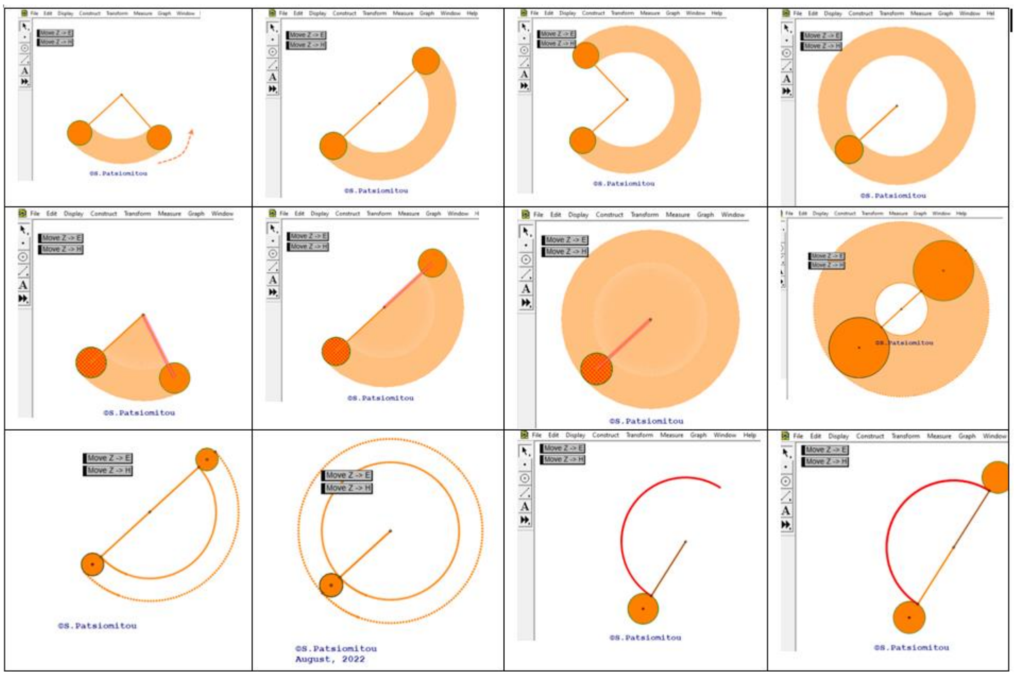


Figure 7. Screenshots of the ball's movement in the DGS environment with the traces thus produced

In figures 8 -10, the circular object (the ball) can go up and down or right and left, simulating the movement of a ball. Traces capture this movement on the screen, providing the visualization of concepts such as right-left and up-down.

The segment (a simulation of a string) can be lengthened and the size of the ball can be transformed. Its qualities (e.g., colour, size) are such as to excite a pupil's curiosity. The task-game *can be used to impress the pupil and increase its gratification while observing the different movements* (Ronge & Ronge, 1858, p. 4).

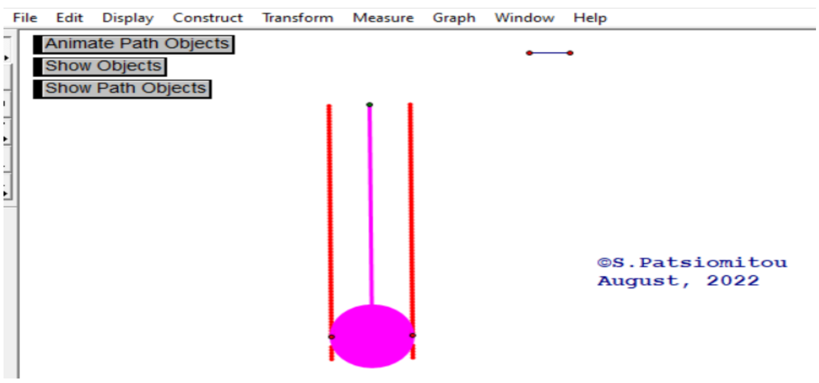


Figure 8. The ball is moving up and down.

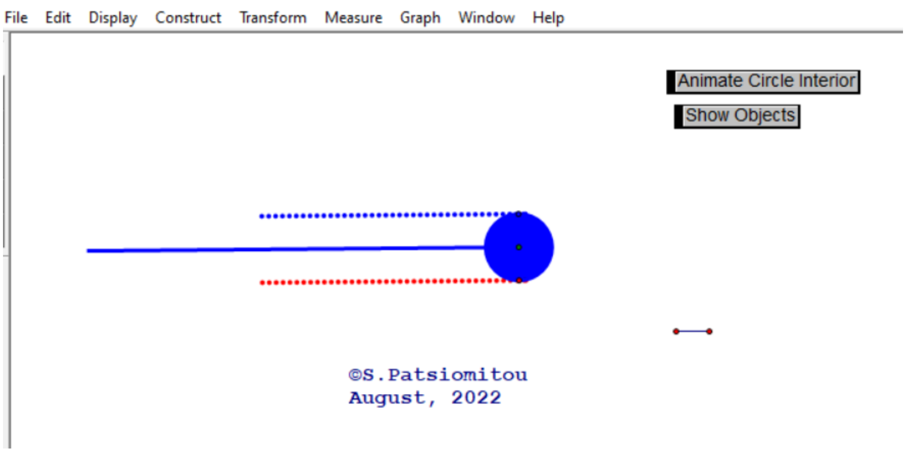


Figure 9. The ball is moving back and forth

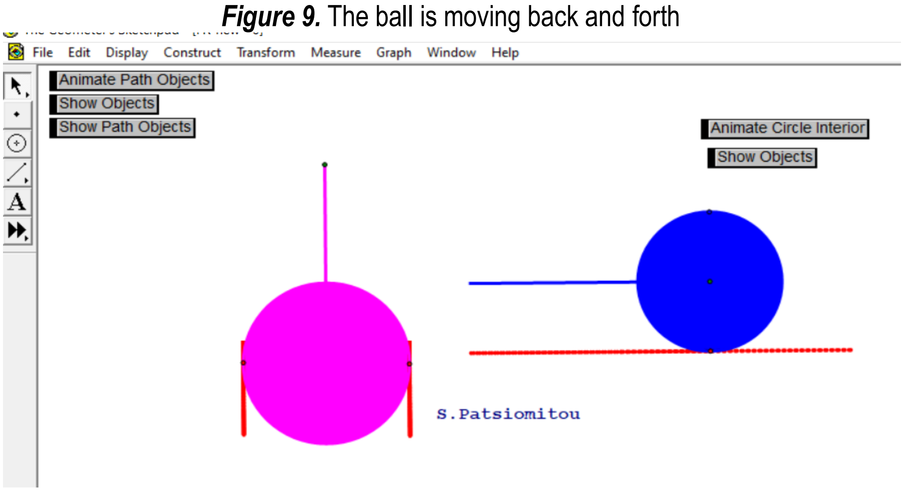
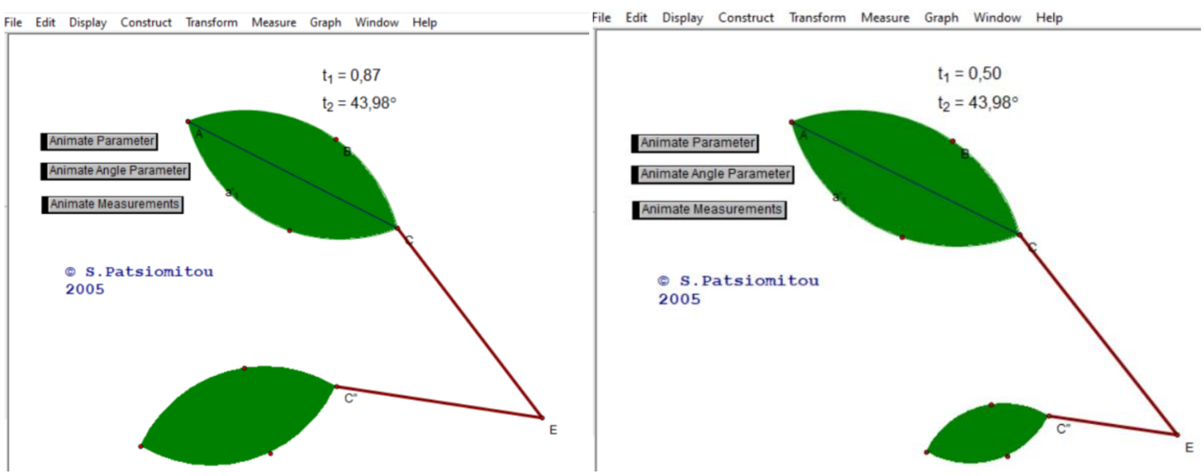


Figure 10. Transforming shapes in the DGS environment

Abraham Arcavi (2003) in his study "*The role of visual representations in the learning of mathematics*" introduces the notion of visualization as a way we can "see" what is unseen. He makes a metaphor and reports examples of the way we can "see" what is unseen through the use of technology (for example, the zoom function or the animation of objects in computer environments is a way to "see" the unseen). According to Arcavi (2003) "In a more figurative and deeper sense, seeing the unseen refers to a more "abstract" world, which no optical or electronic technology can "visualize" for us. Probably, we are in need of a "cognitive technology" (in the sense of Pea, 1987, p. 91) as "*any medium that helps transcend the limitations of the mind … in thinking, learning, and problem-solving activities*." Such "technologies" might develop visual means to better "see" mathematical concepts and ideas" (p.26).



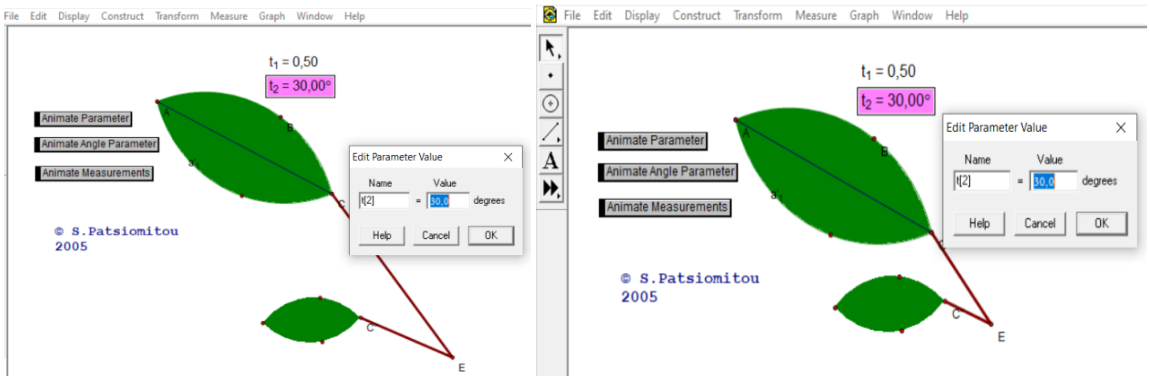


***Figure 11* a,b,c,d.** Playing with “leaves” in GSP (Patsiomitou, 2005, in Greek)

In the GSP file, I have created a simulation of a leaf (Figures 11a, b, c, d) as the resultant of a circular arc and its reflection. By using parameters to scale it, I have reproduced it. We can also move these leaves away from one another or bring one leaf closer to another. Such observations can help a child develop strong experience in developing mathematical concepts and, subsequently, the schemata relating to those concepts. For example: a leaf consists of two circular arcs, the leaves have an axis of symmetry, an angle is formed between the branches of two leaves which is not always the same in all branches, etc.

The modeled tasks are presented in the DGS environment. The pupils will be motivated to manipulate the dynamic objects and to compare them with the real-world objects, physically or in their minds in order “*to bridge connections between the pure world of mathematics --with fixed solutions and “perfect” forms-- and the more messy, ambiguous, or subjective world of experience*” (Sinclair & Jackiw, 2007). In every situation, the experience with a real or simulated object will play a major role for the construction of students’ knowledge. The most important thing is *“the construction of dynamic interdependencies between dynamic objects (or hybrid dynamic objects) diagrams or sections that lies at the heart of the construction of instrumental learning paths* (Patsiomitou, 2021a, p.17). As Stahl (2013)argues “*A major goal of having students experience dynamic geometry is for them to gain an understanding of dynamic dependencies. Dynamic dragging can be used to provide a visual acquaintance with behaviors resulting from hidden dependencies.”* (Stahl, 2013, p.81)

5. Conclusion

The construction of the dynamic circles, semicircles or curves created using the traces in the DGS environment is very important for the *development of schemata* for these concepts. The Fröbelian DGS-Gift can be used by infants, but also by children aged 6-7 in order to extend their conception of the underlined meanings to a higher and more abstract level. It also can be presented in correlation with Gift 11, in which rings, circles and half-circles, or quarter-circles of three different sizes can be used to create several shapes. As Maria–Kraus Boelte& John Kraus (1877) in their book “*The Kindergarten Guide*” report “*after the child has become familiar with the straight line […] the child becomes familiarized with the properties of the curved lines, by laying them in different positions and arranging them in various ways and combinations, thus producing richly varied form”.*

Constantly, changing the diagram through the use of dynamic interaction techniques (Sedig& Sumner, 2006; Sedig, & Liang, 2008; Patsiomitou, 2010) leads to a sequence of geometrically variable transformed diagrams. The use of different techniques in the construction of a geometric dynamic diagram causes different types of interactions. These interactions thus cause different representations on screen, which in turn lead the students to form different mental representations and to translate these verbally in accordance with how they understand them. Steffe&Tzur (1994) in their article "*Interaction and Children's mathem*atics" argued that learning "*occurs as a product of interaction [and] the teacher's interventions is essential in children's learning*." (p. 44). The students' mental representations that take shape can bring about corresponding mental transformations which may stimulate a mental experimental transformation in the diagram, even if it is not realized with material means (i.e., the diagrams on the software screen), but in the student's imagination. Students are frequently able to predict this transformation on the diagram, which they adapt to the particular case.

Through the dynamic diagrams it is important for the pupils to develop a strong intuition of the mathematical concepts that they will learn them later in an abstract level. Following Fischbein (1980) "*an intuitive representation of a phenomenon seems to be […] only a substitute before a complete formal understanding becomes possible*" (p.10).Furthermore, "*play-based math activities have a positive effect on personal-social, fine motor, language, and gross motor developments of children*" (Taner Derman et al., 2020, p. 1). As they argue: "early childhood educators should select purposeful and meaningful learning activities, such as play-based math activities for children" (p. 10).

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