ON DEVELOPMENT OF FOUR-PARAMETERS EXPONENTIATED GENERALIZED EXPONENTIAL DISTRIBUTION

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Abstract
In this paper a four parameter Exponentiated Generalized Exponentiated exponential distribution Is derived from family of GEE and studied. Various properties of the distribution are studied. The distribution is found to be unimodal and has a decreasing and increasing hazard rate depending on the shape parameters. The expressions for the moment, median, quartile, mean deviation, median deviation, skeweness, kurtosis, Renyi entropy are obtained. Several known continuous distributions are found to be special cases of the new distributions. Simulation study and maximum likelihood estimate are used to estimate unknown parameters.

Key words: Moment. Hazard rate. Kurtosis. Renyi Entropy. Unimodal

1 INTRODUCTION

The exponential distribution (ED) also known as negative exponential distribution is a probability distribution that describes the time between event in a Poisson point process i.e a process in which event occurs continuously and independently at a constant average rate. The ED is a very popular statistical model probably, is one of the parametric model most extensively used in several fields; Lemonte et al [1]. The popularity of this distribution can be explained perhaps, by the simplicity of their cumulative function, which involves only one unknown parameter $\lambda > 0$ and takes a simple form $G(x) = 1 - e^{-\lambda x}$ for $x > 0$ in addition to having constant hazard rate.

Gompertz [2] and Verhulst [3,4 and 5] developed several cumulative distribution functions during the first half of the nineteenth century to compare known human mortality tables and represent mortality growth. One of them is as follows

$$G(t) = (1 - \rho e^{-\lambda t})^{\alpha} \ldots\ldots\ldots\ldots 1$$

for $t > {}^{1}/_{\lambda}\, In\rho$. Where ρ, λ *and* α are all positive real numbers. In twentieth century, Ahuja and Nash [6] also considered this model and made some further generalization. The generalized exponential distribution or the exponentiated exponential distribution is defined as a particular case of the Gompertz-Verhulst [2,3,4 and5] distribution function, when $\rho = 1$. Therefore, X is a two parameters generalized exponential random variable if it has the distribution function

$$G(x: \alpha, \lambda) = (1 - e^{-\lambda x})^{\alpha} \ldots\ldots\ldots\ldots\ldots\ldots 2$$

and the density function.

$$g(x: \alpha, \lambda) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1} e^{-xt} \ldots\ldots 3$$

Where α and λ play the role of the shape and scale parameters respectively. Many exponentiated families of distributions have appeared in the literature as generalizations of existing distributions. Mudholkar and Srivastava [7] extended the Weibull distribution by introducing the 3-parameter exponentiated Weibull distribution (EWD) that has bathtub or monotone failure rate. Gupta et al.[8] studied the general properties of the exponentiated families of distributions such as hazard function and some ordering relations. Gupta and Kundu [9] defined a 2-parameter generalized exponential distribution, a particular case of EWD, and studied some of its properties, including hazard rate, moment generating function, distribution of sums and extreme values. They also compared the flexibility of the generalized exponential distribution to a 2-parameter gamma distribution and a 2 parameter Weibull distribution by studying the deep groove ball bearings lifetime data. They concluded that the generalized exponential distribution can be used as alternative to the 2 parameter Weibull distribution and the 2-parameter gamma distribution.

Cadeiro et al [10] proposed a class of distributions by adding two parameters to a continuous distribution, by extending the idea first introduced by Lehman [11] and studied by Nadarah and Kotz [12]. This method leads to a new class of Exponentiated generalized distribution (EG) that can be interpreted as a double construction of Lehmann alternative. The distributions extend the exponentiated type distribution and obtain some of its structure properties. Given a continuous c.d.f. G(x), we define the EG class of distributions by

$$F(x) = [1 - \{1 - G(x)\}^{\alpha}]^{\beta} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 4$$

and

$$f(x) = \alpha\beta\{1 - G(x)\}^{\alpha-1}[1 - \{1 - G(x)\}^{\alpha}]^{\beta-1}g(x) \dots\dots\dots\dots 5$$

Where $\alpha > 0$ and $\beta > 0$ are two additional shape parameters. The EG has tractable properties especially for simulation since its quantile function take a simple form.

$$x = Q_G([1 - (1 - u^{1/\beta})^{1/\alpha}]) \dots..6$$

Where $Q_G(u)$ is the baseline quantile function.

To illustrate the flexibility of EG model, Cordero et al [10] applied EG to some well known distribution such as the Frechet, normal, gamma and Gumbel distributions, with several properties for the EG class, which provide motivations to adopt this generator. The two extra parameters α and β in the density can control both tail weight, and allow generation of flexible distribution, with heavier or lighter tails, as appropriate. There is also an attractive physical interpretation of the EG model when α and β are positive integers see Cordeiro and lemonte [13]. The EG family properties have been explored in recent works. Here, we refer to the papers: Cordeiro et al.[14], Cordeiro and Lemonte [13], Elbatal and Muhammed [15], Oguntunde et al.[16] , da Silva et al.[17] , de Andrade et al [18,19] Cordeiro et al [20] , which used the EG class to extend the Burr III, Birnbaum-Saundersm, inverse Weibull, inverted exponential, generalized gamma , Gumbel ,extended exponential, standardized half-logistic distributions respectively.

Numerous generalized classes of distributions have been developed and applied to explain diverse phenomena. A common feature of these generalized distributions is that they have more parameters. The interests in developing more flexible statistical distribution have remained strong in statistics profession; Alzaatreh et al [21] . Johnson et al.[22] stated that the use of four-parameter distributions should be sufficient for most practical purposes. According to these authors, at least three parameters are needed but they doubted any noticeable improvement arising from including a fifth or sixth parameter. We belief that additional two parameters to an existing exponentiated exponential distribution which serve as alternative distribution to weibull and gamma distribution may generate new distribution with tractable properties. This paper presents yet another four parameters statistical distribution to fit positively skewed distribution.

The rest of the paper is organized as follows. In Section 2 we define the Exponentiated Generalized Exponentiated exponential (EGEE) distribution and outline some special cases of the distribution and how other two existing distributions were formulated in section 3. The graphs of probability density fuction (pdf), cumulative distribution function (cdf) and hazard functions of proposed distribution and other two existing distributions of the family are obtained. In section 4, some mathematical properties and limit behavior are derived, in section 5, the maximum likelihood is used to estimate the unknown parameters, in section 6, we provide some simulated result base on the mathematical properties. We conclude in section 7 base on some significant result on the EGEE distribution.

2.0 Methodology.

Firstly a statistical distribution will be proposed in this study, the distribution would be generated by using exponentiated generalized family frame work with the generalized exponential distribution as the baseline distribution. Secondly, properties of the distribution will be study and the simulation study will be done using R statistics. Thirdly Maximum likelihood estimator will be use to estimate the unknown parameters.

2.1 EXISTING DISTRIBUTIONS FROM EXPONENTIATED GENERALIZED FAMILY

We intend to look at two existing distributions from an Exponentiated Generalized Family of distribution. Cordeiro et al [10] and Oguntunde et al [16] used Exponentiated Generalized Family to add two parameters to Frechet and Inverted Exponential distribution respectively.

2.1.1Exponentiated Generalized Frechet (EGF)

The cdf of the Frechet distribution (for $x > 0$) is $G_{\sigma,\lambda}(x) = \exp\{-(\sigma/x)^{\lambda}\}$, where $\lambda > 0$ and $\sigma > 0$. Then, we defined the Exponentiated Generalized Frechet (EGF) cumulative distibution (for $x > 0$) from (4) as,

$$F(x) = [1 - \{1 - \exp\{-(\sigma/x)^{\lambda}\}\}^{\alpha}]^{\beta} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad 7$$

where $\lambda > 0$, $\sigma > 0$, $\alpha > 0$ and $\beta > 0$. The EGF density function can be obtain from (5) as

$$f(x) = \alpha\beta\lambda\sigma^{\lambda}x^{-(\lambda+1)}\exp\{-(\sigma/x)^{\lambda}\}[1 - \exp\{-(\sigma/x)^{\lambda}\}]^{\alpha-1}(1 - [1 - \exp\{-\left(\frac{\sigma}{x)^{\lambda}}\right\}]^{\alpha})^{\beta-1} \dots\dots 8$$

The hazard function of EGF

$$h(x) = \frac{\alpha\beta\lambda\sigma^{\lambda}x^{-(\lambda+1)}\exp\{-(\sigma/x)^{\lambda}\}[1-\exp\{-(\sigma/x)^{\lambda}\}]^{\alpha-1}(1-[1-\exp\{-(\sigma/x)^{\lambda}\}]^{\alpha})^{\beta-1}}{1-[1-\{1-\exp\{-(\sigma/x)^{\lambda}\}\}^{\alpha}]^{\beta}} \dots\dots\dots\dots 9$$

2.1.2 EXPONENTIATED GENERALIZED INVERTED EXPONENTIAL (EGIE)

The pdf and cdf of the Inverted Exponential (IE) distribution are given respectively by; $g(x) = \frac{\lambda}{x^2}\exp\left(-\frac{\lambda}{x}\right)$ $and\ G(x) = \exp(-\frac{\lambda}{x})$ where $x > 0$, the scale parameter $\lambda > 0$. Hence the EGIE distribution is derived by substituting cdf of IE into equation (4) to obtain

$$F(x) = [1 - \{1 - \exp -(\lambda/x)\}^{\alpha}]^{\beta} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.10$$

The corresponding pdf

$$f(x) = \alpha\beta\lambda x^2 \exp -(\lambda/x)\{1 - \exp -(\lambda/x)\}^{\alpha-1}[1 - \{1 - \exp -\left(\frac{\lambda}{x}\right)\}^{\alpha}]^{\beta-1} \dots.11$$

Where $x > 0, \alpha > 0, \beta > 0, \lambda > 0$

Hazard function of EGIE becomes;

$$h(x) = \frac{\alpha\beta\lambda x^2 \exp -(\lambda/x)\{1-\exp -(\lambda/x)\}^{\alpha-1}[1-\{1-\exp -\left(\frac{\lambda}{x}\right)\}^{\alpha}]^{\beta-1}}{1-[1-\{1-\exp -(\lambda/x)\}^{\alpha}]^{\beta}} \dots\dots\dots\dots\dots..12$$

2.1.3 PROPOSED DISTRIBUTION

Exponentiated Generalized Exponentiated Exponential (EGEE)

The cdf and pdf of the Exponentiated Exponential are presented in equation (13) and

(14) respectively as;

$$G(x) = (1 - e^{-x/\theta})^{k} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots...13$$

$$g(x, \theta, k) = k/\theta(1 - e^{-x/\theta})^{k-1}e^{-\frac{x}{\theta}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots...14$$

Then we defined the Exponentiated Generalized Exponentiated Exponential (EGEE) cumulative distribution from (4) as;

$$F(x) = [1 - \{1 - (1 - e^{-x/\theta})^{k}\}^{\alpha}]^{\beta} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..15$$

By inserting (12) in (4) and the corresponding p.d.f from (23) is

$$f(x) = \frac{\alpha\beta k}{\theta}e^{-\frac{x}{\theta}}(1 - e^{-x/\theta})^{k-1}\{1 - (1 - e^{-x/})^{k}\}^{\alpha-1}[1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]^{\beta-1} \dots.16$$

The hazard function is;

$$h(x) = \frac{\frac{\alpha\beta k}{\theta}e^{-x/\theta}(1 - e^{-x/\theta})^{k-1}\{1 - (1 - e^{-x/\theta})^{k}\}^{\alpha-1}[1 - \{1 - (1 - e^{-x/\theta})^{k}\}^{\alpha}]^{\beta-1}}{1 - [1 - \{1 - (1 - e^{-x/\theta})^{k}\}^{\alpha}]^{\beta}} \dots.17$$

The survival function is;

$$s(x) = 1 - [1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]^{\beta} \dots\dots\dots\dots\dots\dots\dots\dots\dots 18$$

Quantile function of Exponentiated Generalized Exponentiated Exponential (EGEE) is expressed as

$$x = -\theta[ln(1-(1-\{1-u^{\frac{1}{\beta}}\}^{\frac{1}{\alpha}})^{\frac{1}{k}})] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 19$$

where α, β and K are shape parameters while θ is a scale parameter.s

2.1.4 SPECIAL CASE OF EGEE DISTRIBUTION.

- The Exponential distribution is a special case of EGEE when $\alpha = \beta = k = 1$.
- For $\alpha = \beta = 1$ the EGEE gives an Exponentiated Exponential distribution
- When $k = 1$ the EGEE gives a member of Exponentiated Generalized Family which is Exponentiated Generalized Exponential distribution.

3.1 PLOTS OF SOME EXPONENTIATED GENERALIZED FAMILY OF DISTRIBUTIONS

The plots for EGF, EGEE and EGIE distributions for selected parameters value are displayed in figure 1,2,3,4,5,6…9. Figures 1, 2 and 3 are the density function plots for EGEE, EGF and EGIE respectively for various values of the their parameters, Figures 4, 5 and 6 are the cdf plots for EGEE, EGF and EGIE respectively while Figures 7, 8 and 9 are the hazard function plots for EGEE, EGF and EGIE respectively. These plots show that the EGF and EGEE model are fairly flexible and can be used to fit positive skewed data.

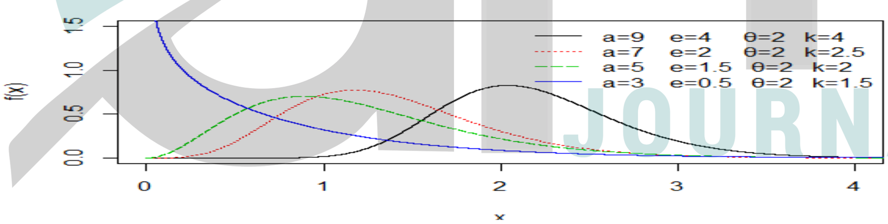

Figure 1 Plot of Density function for EGEE

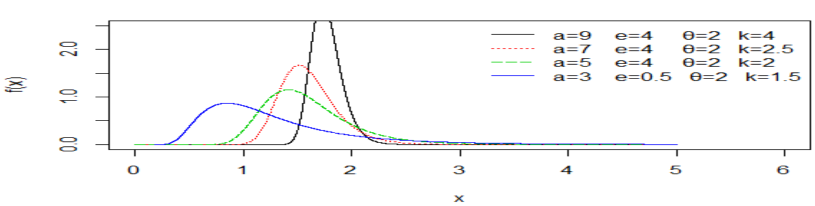

Figure 2 Plot of Density function for EGF

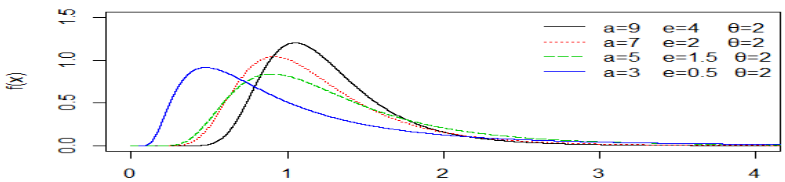

Figure 3 Plot for Density function for EGIE

Figure 1 of EGEE shows values for $\alpha = a, \beta = e, \theta = \theta \; and \; \mathrm{K} = \mathrm{k}$ parameters which are shapes and scale parameters respectively. At $\alpha = 9, \beta = 4, \theta = 2 \; and \; \mathrm{K} = 4$ the graph shows skeweness on both sides, a shift in the graph to left and a peak at 2 in the x axis. At a reduce value of the parameters $\alpha = 7, \beta = 2, \theta = 2 \; and \; \mathrm{K} = 2.5$, the density is right skewed, more close to zero and have a peak between 1 and 2 in the x axis. As the values shape parameter tends to 1, skewness at left side tend to disappear and becomes heavier at the right side of the graph. The last graph on figure 1 shows a decrease (inverted J shape) as the shape parameter values are less than 1.

Figure 2 from EGF with the same numbers of shapes and scale parameters show high peak for a high values of shape parameter. The peak tends to reduce as the shapes parameter reduces. They both exhibit similar skewness level but have different starting point on the plots. Figure 3 shows plot of EGIE distribution for shapes and scale parameters. The origin of graphs and the projection point came earlier than the other two graphs. The peaks for the graphs did not behave in accordance with the size of parameters.

The graphs for EGEE shows the flexibility of our proposed distribution with the following attribute: uni-modal, decreasing (inverted J shape), increasing and can be skewed at both end depending on the values of the parameter. There are strong influence of values shape parameter on EGEE distribution than EGF and EGIE.

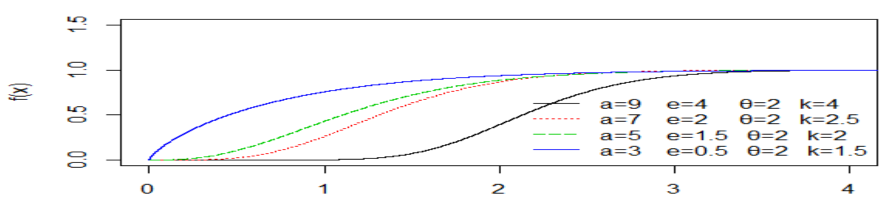

Figure 4 Plot of Cumulative Distribution Function (cdf) for EGEE

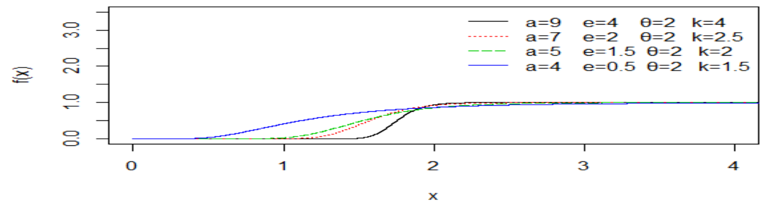

Figure 5 Plot of Cumulative Distribution Function (cdf) for EGF

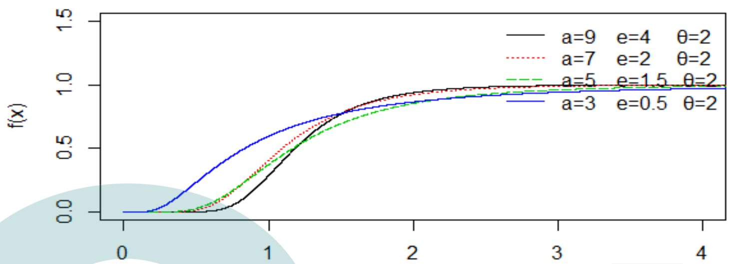

Figure 6 Plot of Cumulative Distribution Function (cdf) for EGIE

Figure 4 of EGEE distribution shows true nature of cdf of no plot exceeding the bench mark of 1. For a high values of shape parameter (At $\alpha = 9, \beta = 4, \theta = 2$ *and* $\mathrm{K} = 4$) the graph moved at a constant rate from point zero to a value close to two on the x-axis before projecting to one on the y-axis. As the shapes parameters decrease the projection point on the graph reduced to a value lesser than one on the x axis. At shapes parameter of $\alpha = 3, \beta = 0.5$ *and* $\mathrm{K} = 1.5$ the graph made no movement on the x axis.

Figure 5 of EGF cdf shows a similar pattern like that of EGEE for shapes and scale parameters $\alpha = 9, \beta = 4, \theta = 2$ *and* $\mathrm{K} = 4$, as the shapes parameters decrease EGF exhibit the same kind of pattern in respective of the values of the parameters.

Figure 6 of EGIE cdf with less shapes parameters shows different pattern of movement compare to EGEE and EGF. For high values of shapes and scale parameters $\alpha = 9, \beta = 4,$ *and* $\theta = 2,$ the movement on the x axis tend to move close to value one before projecting to one on the y axis, as shape parameter reduce the movement on the x axis reduces.

From figures 4,5 and 6 we can conclude that the effect of shapes parameters has more significant influence on EGEE distribution than EGF and EGIE distributions.

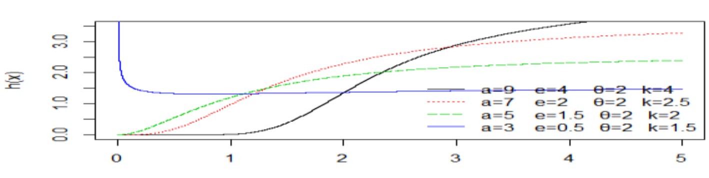

Figure7 Plot of Hazard function for EGEE

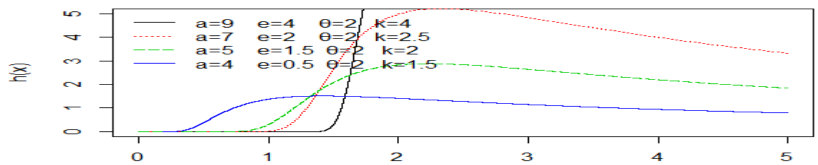

Figure8 Plot of Hazard function for EGF

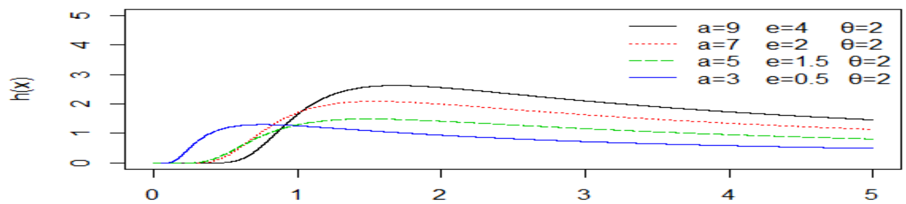

Figure 9 Plot of Hazard function for EGIE

Figure 7 shows EGEE distributions can have constant, increasing or decreases, depending on the shapes and scale parameters. For high values of shapes parameters$\alpha = 9, \beta = 4$ *and* K $= 4$ the hazard function tend to be constant at a point before increasing, as the shapes parameters decrease ,there was a sharp increase in the hazard function before a slow movement at a later stage, while for values of shape parameter less than one ($\alpha = 3, \beta = 0.5$ *and* K $= 1.5$) the shape of hazard is decreasing and constant.

Figure 8 of EGF shows hazard graph which can be decreasing, increasing and constant depending on the shapes parameter. While Figure 9 of EGIE distribution has a decreasing and increasing hazard plot.

From figure 7, 8 and 9 the EGEE graph seems to be more sensitive to change in parameter values.

4.1 PROPERTIES OF EXPONENTIATED GENERALIZED EXPONENTIATED EXPONENTIAL (EGEE)

The cumulative and the probability distribution of Exponentiated Exponential are expressed respectively in equation 12 and 13 as.

$$G(x) = (1 - e^{-\frac{x}{\theta}})^k$$

$$g(x, \theta, k) = \frac{k}{\theta}(1 - e^{-\frac{x}{\theta}})^{k-1} e^{-\frac{x}{\theta}}$$

The properties of the proposed distribution will be derived from Exponentiated Generalized Family (EGF) distribution in equation 4 and 5.

$$F(x) = [1 - \{1 - G(x)\}^\alpha]^\beta$$

$$\boldsymbol{\alpha} > 0, \beta > 0, x \epsilon \mathbb{R}$$

$$f(x) = \alpha\beta\{1 - G(x)\}^{\alpha-1}[1 - \{1 - G(x)\}^{\alpha}]^{\beta-1} g(x)$$

Expansion for the density function

$$(\mathbf{1} - \boldsymbol{z})^{\boldsymbol{\beta}-\mathbf{1}} = \sum_{\boldsymbol{\lambda}=\mathbf{0}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{\lambda}} \boldsymbol{\Gamma}(\boldsymbol{\beta})}{\boldsymbol{\Gamma}(\boldsymbol{\beta} - \boldsymbol{\lambda})\boldsymbol{\lambda}!} \boldsymbol{z}^{\boldsymbol{\lambda}} \quad |z|<1, \beta \epsilon \mathbb{R} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{20}$$

Thus using similar expansion on equation 15

$$F(x) = \sum_{\boldsymbol{\lambda}=\mathbf{0}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{\lambda}} \boldsymbol{\Gamma}(\boldsymbol{\beta} + \mathbf{1})}{\boldsymbol{\Gamma}(\boldsymbol{\beta} - \boldsymbol{\lambda} + \mathbf{1})\boldsymbol{\lambda}!} \{1 - G(x)\}^{\alpha\lambda}$$

Consider

$$\{1 - G(x)\}^{\alpha\lambda} = \sum_{\boldsymbol{j}=\mathbf{0}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{j}} \boldsymbol{\Gamma}(\alpha\lambda + \mathbf{1})}{\boldsymbol{\Gamma}(\alpha\lambda - \boldsymbol{j} + \mathbf{1})\, \boldsymbol{j}!} G(x)^{\boldsymbol{j}}$$

$$F(x) = \sum_{j=0}^{\infty} \sum_{\boldsymbol{\lambda}=\mathbf{0}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{\lambda}+\boldsymbol{j}} \boldsymbol{\Gamma}(\boldsymbol{\beta} + \mathbf{1}) \boldsymbol{\Gamma}(\alpha\lambda + \mathbf{1})}{\boldsymbol{\Gamma}(\boldsymbol{\beta} - \boldsymbol{\lambda} + \mathbf{1}) \boldsymbol{\Gamma}(\alpha\lambda - \boldsymbol{j} + \mathbf{1})\, \boldsymbol{j}!\, \boldsymbol{\lambda}!} G(x)^{\boldsymbol{j}} \ldots\ldots\ldots \mathbf{21}$$

$$F(x) = \sum_{j=0}^{\infty} \boldsymbol{w}_{\boldsymbol{j}}\, G(x)^{\boldsymbol{j}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{22}$$

Where $w_j = w_j(\alpha, \beta) = \sum_{\boldsymbol{\lambda}=\mathbf{0}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{\lambda}+\boldsymbol{j}} \boldsymbol{\Gamma}(\boldsymbol{\beta}+\mathbf{1}) \boldsymbol{\Gamma}(\alpha\lambda+\mathbf{1})}{\boldsymbol{\Gamma}(\boldsymbol{\beta}-\boldsymbol{\lambda}+\mathbf{1}) \boldsymbol{\Gamma}(\alpha\lambda-\boldsymbol{j}+\mathbf{1})\, \boldsymbol{j}!\boldsymbol{\lambda}!}$

Differentiating equation 21 with respect to x gives the pdf

$$f(x) = \sum_{j=0}^{\infty} j \boldsymbol{w}_{\boldsymbol{j}} \boldsymbol{g}(\boldsymbol{x}) \left(G(x)\right)^{\boldsymbol{j}-\mathbf{1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots.. \quad \ldots \mathbf{23}$$

Putting equation 13 and 14 into equation 22

$$f(x) = \sum_{j=0}^{\infty} j \boldsymbol{w}_{\boldsymbol{j}} \frac{k}{\theta} (1 - e^{-\frac{x}{\theta}})^{k-1} e^{-\frac{x}{\theta}} (1 - e^{-\frac{x}{\theta}})^{k(j-1)}$$

$$f(x) = \frac{k}{\theta} \sum_{j=0}^{\infty} j \boldsymbol{w}_{\boldsymbol{j}}\, e^{-\frac{x}{\theta}} (1 - e^{-\frac{x}{\theta}})^{kj-1} \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots 24$$

4.1.1 THE r^{th} MOMENT OF PROPOSED EGEE DISTRIBUTION

$$\mu^r = \int_0^\infty x^r f(x)\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \ \ldots\ldots\ldots\ldots\ldots..25$$

Using the expansion of equation 11

$$\mu^r = \int_0^\infty x^r \frac{k}{\theta}\sum_{j=0}^{\infty} j w_j\, e^{-\frac{x}{\theta}}(1-e^{-\frac{x}{\theta}})^{kj-1}\,dx$$

$$\mu^r = \frac{k}{\theta}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)\,m!} j w_j \int_0^\infty \left(\frac{\theta y}{m+1}\right)^r e^{-x\left(\frac{m+1}{\theta}\right)}\frac{\theta dy}{m+1}$$

$$\mu^r = \frac{k\theta^r}{(m+1)^{r+1}}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)\,m!} j w_j\ \Gamma(r+1) \ldots\ldots\ldots\ldots\ldots 26$$

The mean of the proposed EGEE distribution is gotten by making μ^r moment equal to one (|r=1)

$$\mu_/^1 = \frac{k\theta}{(m+1)^2}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)\,m!} j w_j \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..27$$

$$\sigma^2 = \frac{k\theta^2}{(m+1)^3}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)\,m!} j w_j \left(2-\frac{k}{(m+1)}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)\,m!} j w_j\right)\ldots\ldots.30$$

4.1.3 MEDIAN OF EGEE DISTRIBUTION

$$F(x) = p_r(X \le m) = \int_0^m f(x)dx = 0.5 \ldots\ldots\ldots.31$$

The median of EGEE Distribution can be obtain by equating equation 14 to 0.5

$$F(x) = [1-\{1-(1-e^{-\frac{x}{\theta}})^k\}^\alpha]^\beta$$

$$x = -\theta[ln(1-\left(1-\left\{1-(0.5)^{\frac{1}{\beta}}\}^{\frac{1}{\alpha}}\right)^{\frac{1}{k}}\right)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots.32$$

4.1.4 MOMENT GENERATING FUNCTION OF EGEE DISTRIBUTION

$$M_x(t) = E(e^{tx})$$

$$E(e^{tx}) = \int_{0}^{\infty} e^{tx} \boldsymbol{f(x)\, dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \ \ldots\ldots\ldots\ldots\ldots\ldots \mathbf{33}$$

$$\int_{0}^{\infty} e^{tx} \frac{k}{\theta} \sum_{j=0}^{\infty} j\boldsymbol{w_j}\, e^{-\frac{x}{\theta}} (1 - e^{-\frac{x}{\theta}})^{kj-1} \boldsymbol{dx}$$

$$M_x(t) = \frac{k}{(m+1-\theta t)} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \binom{kj-1}{m} j\boldsymbol{w_j} \ldots\ldots\ldots\ldots.34$$

$$\boldsymbol{\mu}_{/}^{\mathbf{1}} = M^{/}{}_x(t)$$

$$M^{/}{}_x(0) = \frac{\theta k}{(m+1)^2} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \binom{kj-1}{m} j\boldsymbol{w_j}$$

4.1.5 ORDER STATISTICS OF EXPONENTIATED GENERALIZED FAMILY DISTRIBUTION

The density $f_{n:i}(x)$ of the ith order statistics, for $i = 1, \ldots\ldots n$, from independent identical distribution random variable $Y_1 \ldots\ldots Y_n$ is given by

$$f_{n:i}(x) = \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1} \left(1 - F(x)\right)^{n-i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 35$$

Substitute in equation 35 for pdf and cdf EGF distribution

$$f_{n:i}(x) = \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1} (1 - F(x))^{n-i}$$

$$f_{n:i}(x) = \frac{\propto \beta g(x)(1-G(x))^{\alpha-1}(1-(1-G(x))^{\alpha})^{\beta-1}}{B(i, n-i+1)} (1-(1-G(x))^{\alpha})^{\beta(i-1)} \times \left(1-(1-(1-G(x))^{\alpha})^{\beta}\right)^{n-i}$$

$$f_{n:i}(x) = \frac{\propto \beta g(x)(1-G(x))^{\alpha-1}}{B(i, n-i+1)} (1-(1-G(x))^{\alpha})^{\beta i-1} \times \left(1-(1-(1-G(x))^{\alpha})^{\beta}\right)^{n-i}$$

$$f_{n:i}(x) = \frac{\propto \beta g(x)}{B(i, n-i+1)} \sum_{q=0}^{n-i} \sum_{p=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{q+p+l} \binom{n-i}{q} \binom{\beta(q+i)-1}{p} \binom{\alpha(p+1)-1}{l} G(x)^l$$

Let $s_l = \sum_{q}^{n-i} \sum_{p}^{\infty} (-1)^{q+p+l} \binom{n-i}{q} \binom{\beta(q+i)-1}{p} \binom{\alpha(p+1)-1}{l}$

$$\frac{\propto \beta}{B(i,n-i+1)}\sum_{l=o}^{\infty} s_l g(x)G(x)^l \ldots\ldots\ldots\ldots .36A$$

Order statistics of EGEE distribution is obtain by replacing the cdf and pdf of exponentiated exponential in equation 36A

$$\frac{\propto \beta}{B(i,n-i+1)}\sum_{l=o}^{\infty} s_l \frac{k}{\theta} e^{-\frac{x}{\theta}}\left(1-e^{-\frac{x}{\theta}}\right)^{k-1}\left(\left(1-e^{-\frac{x}{\theta}}\right)^{(k-1)}\right)^{l}$$

$$\frac{\propto \beta}{B(i,n-i+1)}\frac{k}{\theta}\sum_{l=o}^{\infty}\sum_{d=0}^{\infty}(-1)^d \frac{\Gamma(\mathrm{k}(l+1)-l)}{\Gamma(\mathrm{k}(l+1)-l-d)\mathrm{d}!} s_l e^{-x\left(\frac{d+1}{\theta}\right)}$$

Hence the order statistics for EGEE distribution is

$$\frac{\propto \beta}{B(i,n-i+1)}\frac{k}{\theta}\sum_{q=0}^{n-i}\sum_{p=0}^{\infty}\sum_{l=o}^{\infty}\sum_{d=0}^{\infty}(-1)^{p+q+l+d} \frac{\Gamma(\mathrm{n-i+1})\Gamma(\beta(\mathrm{q+i})\Gamma\alpha(\mathrm{p+1})\Gamma(\mathrm{k}(l+1)-l)}{\Gamma(\mathrm{n-i+q+1})\Gamma(\beta(\mathrm{q+i})-\mathrm{p})\Gamma(\alpha(\mathrm{p+1})-l)\Gamma(\mathrm{k}(l+1)-l-$$

The above expression is equation 36B

4.1.6 SKEWNESS AND KURTOSIS OF THE EGEE DISTRIBUTION

The skewness and kurtosis of the EGEE distribution shall be examined using two approaches. These approaches include the measure of skewness(s.k) and kurtosis(k.u) based on moments and the measure of skewness and kurtosis based on quantiles. In the moments based approach,

$$S.K = \frac{\mu_3^{/} - 3\mu\mu_2^{/} + 2\mu^2}{\left(\mu_2^{/} - \mu^2\right)^{3/2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 37$$

And

$$K.U = \frac{\mu_4^{/} - 4\mu\mu_3^{/} + 6\mu^2\mu_2^{/} - 3\mu^4}{\left(\mu_2^{/} - \mu^2\right)^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .38$$

The quantile measure based approach of evaluating skewness and kurtosis of a distribution is particularly useful when the quantile function of a distribution exists in closed form or in a simple analytic expression. Galton [23] proposed a quantile measure based approach for evaluating skewness while Moor [24] did the same for Kurtosis. Galton's skewness and Moor's kurtosis is evaluated using the relations

$$S.K = \frac{\mathrm{Q}(6/8) - 2\mathrm{Q}(4/8) + \mathrm{Q}(3/8) + \mathrm{Q}(2/8)}{\mathrm{Q}(6/8) - \mathrm{Q}(2/8)} \ldots\ldots\ldots .39$$

$$K.U = \frac{\mathrm{Q}(7/8) - 2\mathrm{Q}(5/8) + \mathrm{Q}(3/8) + \mathrm{Q}(1/8)}{\mathrm{Q}(6/8) - \mathrm{Q}(2/8)} \ldots\ldots\ldots\ldots 40$$

Since the Quantile function of the EGEE distribution exists in closed form as given in (18), then (39) and (40) can be used in evaluating the skewness and kurtosis of the EGEE Distribution.

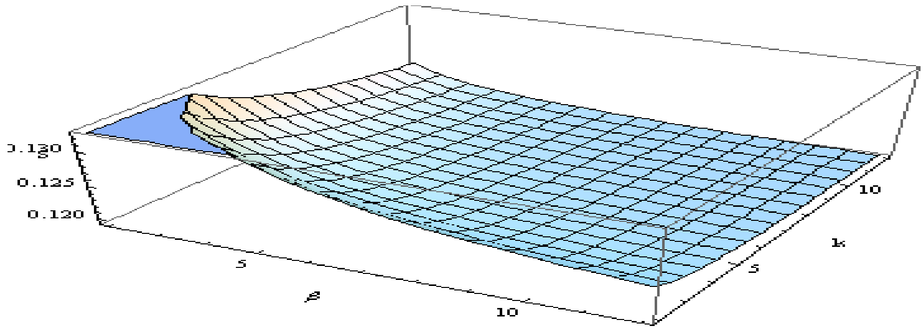

Figure 10 for EGEE Skewness

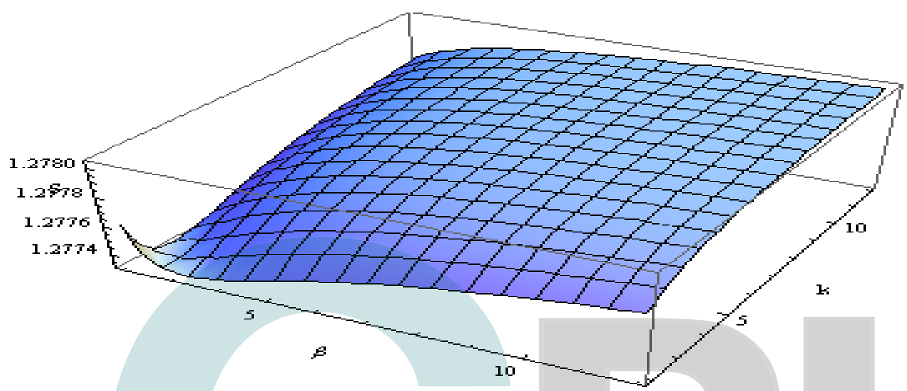

Figure 11 for EGEE Kurtosis

The 3D plot for skewness and kurtosis were plotted using the quartile function of the EGEE distribution with $\alpha = \theta = 1$ while $\beta = k$ takes values from 2 to 12**.**

4.1.7 THE MEAN DEVIATION

The deviation from the mean (in the case of the symmetric distributions) or the deviation from the median (in the case of skewed distributions) can be used as a measure of spread in the population. Let X be a EGEE random variable with mean $\mu = E(X)$ and median M. The mean deviation ($D(\mu)$) from the mean and the mean deviation ($D(M)$) from the median are defined respectively by

$$D(\mu) = E\{|X-\mu|\} = \int_{-\infty}^{\infty} |x-\mu| f_x dx \ldots\ldots\ldots\ldots\ldots\ldots 41$$

$$= \int_{-\infty}^{\mu} (\mu - x) f_x dx + \int_{\mu}^{\infty} (x-\mu) f_x dx$$

$$2\mu F_x(\mu) - 2\int_{-\infty}^{\mu} x f_x(x) dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 42$$

Where $\int_{-\infty}^{\mu} x f_x(x) dx$ and $F_x(\mu)$ are incomplete moment and cumulative function respectively.

For EGEE distribution

$$D(\mu) = 2\mu F_x(\mu) - 2\int_{0}^{\mu} x f_x(x) dx \ldots\ldots\ldots\ldots\ldots 43$$

$F_x(\mu)$is obtained from equation 33 as $F(u) = [1-\{1-(1-e^{-\mu/\theta})^k\}^\alpha]^\beta$ and $\int_0^\mu x f_x(x) dx$ is obtain from equation 43 as $\frac{k\theta}{(m+1)^2}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)\,m!} j w_j\ \gamma(2,\mu)$

Where $\gamma(2,\mu)$ is lower incomplete gamma function.

$$D(M) = E\{|X-\mu|\} = \int_{-\infty}^{\infty} |x-M| f_x dx \dots\dots\dots 44$$

$$= \int_{-\infty}^{M} (M-x) f_x dx + \int_{M}^{\infty} (x-M) f_x dx$$

$$\mu - 2\int_{-\infty}^{M} x f_x(x) dx \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..45$$

$\int_0^M x f_x(x) dx$ is obtain from equation 43 as $\frac{k\theta}{(m+1)^2} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(kj)}{\Gamma(kj-m)\, m!} j w_j \ \gamma(2, \mathrm{M})$

4.1.8 ASYMPTOTIC BEHAVIOR

We seek to investigate the behavior of the proposed model as given in Equation 34 as $x \to 0$ and as $x \to \infty$. This involves considering $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left[\frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}} (1-e^{-x/\theta})^{k-1} \{1-(1-e^{-x/})^k\}^{\alpha-1} [1-\{1-(1-e^{-\frac{x}{\theta}})^k\}^{\alpha}]^{\beta-1}\right]$$
$$= 0$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \left[\frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}} (1-e^{-x/\theta})^{k-1} \{1-(1-e^{-x/})^k\}^{\alpha-1} [1-\{1-(1-e^{-\frac{x}{\theta}})^k\}^{\alpha}]^{\beta-1}\right]$$
$$= 0$$

These results confirm further that the proposed distribution has a mode ;Oguntunde el at [16].

4.1.9 RENYI ENTROPY

The entropy of X is a measure of variation of the uncertainty. There are many entropy measures studied and discussed in literature but the Renyi entropy is perhaps one of the most popular. The Renyi entropy of X with EGEE density is given by

$$I_{R(\rho)} = \frac{1}{(1-\rho)} log\left(\int_0^{\infty} f(x)^{\rho} dx\right) \dots\dots\dots\dots\dots\dots..46$$

$$where\ \rho > 0\ and\ \rho \neq 1$$

$$I_{R(\rho)} = \frac{1}{(1-\rho)} log\left(\frac{k^{\rho}}{\theta^{\rho}} \sum_{j=0}^{\infty} (j\boldsymbol{w_j})^{\rho} \int_0^{\infty} e^{-\frac{\rho x}{\theta}} (1-e^{-\frac{x}{\theta}})^{\rho(kj-1)} dx\right)$$

$$I_{R(\rho)} = \frac{1}{(1-\rho)} log\left(\frac{k^{\rho}}{\theta^{\rho}} \sum_{j=0}^{\infty} \sum^{\infty} \quad (j\boldsymbol{w_j})^{\rho} \binom{\rho(kj-1)}{b} (-1)^b \frac{1}{(\rho+b)}\right) \dots\dots\dots..47$$

5. 1 MAXIMUM LIKELIHOOD.

In this section we determine the maximum likelihood estimates (MLEs) of the parameters of the EGEE distribution. For a random sample $x_1 x_2 \dots . x_n$ of size n, the log-likelihood function of 4 parameter EGEE distribution is given by

$$L = \sum_{i=1}^{n} \ln(fx)$$

$$= \sum_{i=1}^{n} ln\left(\frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}}(1 - e^{-x/\theta})^{k-1}\{1 - (1 - e^{-x/\theta})^{k}\}^{\alpha-1}[1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]^{\beta-1}\right)$$

$$nln\alpha + nln\beta + nlnk - nln\theta - \sum_{i=1}^{n} \frac{x}{\theta} + (k - 1)\sum_{i=1}^{n} \ln\left(1 - e^{-\frac{x}{\theta}}\right) + (\alpha - 1)\sum_{i=1}^{n} \ln\{1 - (1 - e^{-\frac{x}{\theta}})^{k}\} + (\beta - 1)\sum_{i=1}^{n} \ln[1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\{1 - (1 - e^{-\frac{x}{\theta}})^{k}\} + \frac{(\beta - 1)\sum_{i=1}^{n} \quad [1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]ln\{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}}{[1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln[1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]$$

$$\frac{\partial L}{\partial k} = \frac{n}{k} + \sum_{i=1}^{n} \ln\left(1 - e^{-\frac{x}{\theta}}\right) - (\alpha - 1)\sum_{i=1}^{n} \frac{(1 - e^{-\frac{x}{\theta}})^{k} ln\left(1 - e^{-\frac{x}{\theta}}\right)}{\{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}} + \alpha(\beta - 1)\sum_{i=1}^{n} \frac{\{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha-1}(1 - e^{-\frac{x}{\theta}})^{k} ln\left(1 - e^{-\frac{x}{\theta}}\right)}{[1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]}$$

$$\frac{\partial L}{\partial \theta} = \frac{-n}{\theta} + \sum_{i=1}^{n} \frac{x}{\theta^2} - (k + 1)\sum_{i=1}^{n} \frac{xe^{-\frac{x}{\theta}}}{\theta^2\left(1 - e^{-\frac{x}{\theta}}\right)} + (\alpha - 1)\sum_{i=1}^{n} \frac{k(1 - e^{-\frac{x}{\theta}})^{k-1} xe^{-\frac{x}{\theta}}}{\theta^2\left(1 - \left\{1 - e^{-\frac{x}{\theta}}\right\}^{k}\right)} - (\beta - 1)\sum_{i=1}^{n} \frac{\alpha k(1 - e^{-\frac{x}{\theta}})^{k-1} xe^{-\frac{x}{\theta}}\{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha-1}}{\theta^2\left(1\left(-1 - \left\{1 - e^{-\frac{x}{\theta}}\right\}^{k}\right)^{\alpha}\right)}$$

Solving the nonlinear system of equation of $\frac{\partial L}{\partial \alpha} = 0, \frac{\partial L}{\partial \beta} = 0, \frac{\partial L}{\partial k} = 0$ and $\frac{\partial L}{\partial \theta} = 0$ gives the maximum likelihood estimates of α, β, k and θ respectively. We obtain the 4×4 observed information matrix through,

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{k}\\ \hat{\theta}\end{pmatrix}\left[\begin{pmatrix}\alpha\\ \beta\\ k\\ \theta\end{pmatrix}\begin{pmatrix}\hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha k} & \hat{V}_{\alpha\theta}\\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta k} & \hat{V}_{\beta\theta}\\ \hat{V}_{k\alpha} & \hat{V}_{k\beta} & \hat{V}_{kk} & \hat{V}_{k\theta}\\ \hat{V}_{\theta\alpha} & \hat{V}_{\theta\beta} & \hat{V}_{\theta k} & \hat{V}_{\theta\theta}\end{pmatrix}\right]$$

$$V^{-1} = -E\begin{pmatrix}V_{\alpha\alpha} & V_{\alpha\beta} & V_{\alpha k} & V_{\alpha\theta}\\ V_{\beta\alpha} & V_{\beta\beta} & V_{\beta k} & V_{\beta\theta}\\ V_{k\alpha} & V_{k\beta} & V_{kk} & V_{k\theta}\\ V_{\theta\alpha} & V_{\theta\beta} & V_{\theta k} & V_{\theta\theta}\end{pmatrix}$$

Where

$$V_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}, \quad V_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}, \quad V_{kk} = \frac{\partial^2 L}{\partial k^2}, \quad V_{\theta\theta} = \frac{\partial^2 L}{\partial\theta^2}$$

$$V_{\alpha\beta} = V_{\beta\alpha} = \frac{\partial^2 L}{\partial\alpha\beta}, V_{\alpha k} = V_{k\alpha} = \frac{\partial^2 L}{\partial\alpha k}, V_{\alpha\theta} = V_{\theta\alpha} = \frac{\partial^2 L}{\partial\theta\alpha},$$

$$V_{\beta k} = V_{k\beta} = \frac{\partial^2 L}{\partial\beta k}, \quad V_{\beta\theta} = V_{\theta\beta} = \frac{\partial^2 L}{\partial\theta\beta}, V_{k\theta} = V_{\theta k} = \frac{\partial^2 L}{\partial\theta k}$$

The solution to the above inverse dispersion matrix yields the asymptotic variance and covariance of the maximum likelihood estimators $\hat{\alpha}\ \hat{k}\ \hat{\beta}$ and $\hat{\theta}$. The confidence interval for $\alpha, \beta, k\ and\ \theta$ is given by

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{\alpha\alpha}}, \qquad \hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{\beta\beta}}, \hat{k} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{kk}}, \hat{\theta}, \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{\theta\theta}}$$

Where $Z_{\frac{\alpha}{2}}$ is the α^{th} percentiles of the standard normal distribution.

6.1 SIMULATION STUDY

Simulation study is conducted using the quartile function of the EGEE distribution in equation 19 with the help of R-statistics package, a sample size n=10 is used with different shape and scale parameters combination. Table 1; contain the mean, standard deviation and median of the 4-parameter EGEE distribution for different parameter values. In Table2, some values of Skewness and Kurtosis are obtained with the same combination of parameters

Table 1: The mean ,standard deviation and mean deviation of EGEE distribution for ϴ = 2,4 and 6

β	α	K	ϴ=2			ϴ=4			ϴ=6		
			Mean	SD	MD	Mean	SD	MD	Mean	SD	MD
0.5	0.5	0.5	0.9881	1.1960	0.9494	1.9763	2.3920	1.8988	2.9645	3.5880	2.8483
		0.8	1.4591	1.4541	1.5719	2.9182	2.9095	3.1438	4.3772	4.3642	4.7158
		5	4.0798	2.3439	4.7558	8.1596	4.6879	9.5117	12.2394	7.0317	14.2675
		8	4.9063	2.4738	5.6598	9.8126	4.9477	11.3196	14.7189	7.4215	16.9794
	0.8	0.5	1.7873	1.6779	1.9866	3.5747	3.3557	3.9731	5.3619	5.0336	8.9649
		0.8	2.3864	1.9472	2.7638	4.7729	3.8943	5.5276	7.1592	5.8416	8.2914
		5	5.3494	2.6859	6.1927	10.6988	5.3719	12.3854	16.0482	8.0577	18.5782
		8	6.2219	2.7717	7.1155	12.4438	5.5433	14.2309	18.6656	8.3149	21.3464
	5	0.5	6.9609	3.3771	8.1485	13.9219	6.7543	16.2971	20.8829	10.1314	24.4456
		0.8	7.8523	3.4522	9.0821	15.7045	6.9045	18.1642	23.5568	10.3567	27.2464

		5	11.4456	3.5659	12.7383	22.8912	7.1319	25.4766	34.3369	10.6978	38.2149
		8	12.3803	3.5744	13.6777	24.7607	7.1489	27.3553	37.141	10.7233	41.0329
	8	0.5	8.6479	3.5899	9.9517	17.2959	7.1798	19.9033	25.9437	10.7695	29.855
		0.8	9.5646	3.6279	10.8891	19.1292	7.2559	21.7782	28.6938	10.8839	32.6672
		5	13.1962	3.6832	14.5506	26.3925	7.3663	29.1012	39.5887	11.0495	43.6518
		8	14.1338	3.6871	15.4904	28.2676	7.3744	30.9807	12.4015	11.0616	46.4711
5	0.5	0.5	0.0219	0.0341	0.0166	0.0437	0.0683	0.0333	0.0656	0.1024	0.0499
		0.8	0.0988	0.1052	0.1025	0.1977	0.2104	0.205	0.2966	0.3156	0.3075
		5	1.6329	0.7460	1.9305	3.2659	1.4920	3.8609	4.8989	2.2382	5.7914
		8	2.3349	0.8861	2.7023	4.6698	1.7721	5.4046	7.0047	2.6582	8.1069
	0.8	0.5	0.0449	0.0540	0.0431	0.0899	0.1081	0.0861	0.1349	0.1621	0.1292
		0.8	0.1672	0.1449	0.1891	0.3344	0.2898	0.3781	0.5016	0.4346	0.5672
		5	1.9778	0.7475	2.2819	3.9555	1.4951	4.5638	5.9332	2.2426	6.8457
		8	2.7302	0.8514	3.0851	5.4603	1.7028	6.1701	8.1905	2.5542	9.2552
	5	0.5	0.2658	0.1644	0.3100	0.5316	0.3288	0.6200	0.7973	0.4932	0.9301
		0.8	0.6080	0.2884	2.1169	1.2160	0.5769	1.4113	1.8241	0.8654	2.1169
		5	3.2184	0.6224	3.4699	6.4368	1.2447	6.9398	9.6552	1.8671	10.4097
		8	9.6552	1.8671	10.4097	8.1439	1.3114	8.6781	12.2159	1.9671	13.0171
	8	0.5	0.3567	0.1928	0.4144	0.7134	0.3856	0.8289	1.0701	0.5784	1.2434
		0.8	0.7549	0.3111	0.8642	1.5098	0.6223	1.7284	2.2647	0.9334	2.5926
		5	3.5005	0.5911	3.7378	7.0001	1.1829	7.4756	10.5014	1.7734	11.2135
		8	4.3669	0.6174	4.6174	8.7339	1.2348	9.2329	13.1008	1.8522	13.8492

Table2 : The Skewness and Kurtosis of EGEE distribution for Θ = 2,4 and 6

β	α	k	ϴ=2		ϴ=4		ϴ=6	
			SK	KT	SK	KT	SK	KT
0.5	0.5	0.5	1.5700	1.6535	1.5700	1.6535	1.5700	1.6535
		0.8	1.1411	0.6917	1.1411	0.6917	1.1411	0.6917
		5	0.1097	-0.7757	0.1097	-0.7757	0.1097	-0.7757
		8	0.0036	-0.8161	0.0036	-0.8161	0.0036	-0.8161
	0.8	0.5	0.9649	0.3352	0.9649	0.3352	0.9649	0.3352
		0.8	0.6261	-0.2490	0.6261	-0.2490	0.6261	-0.2490
		5	-0.0394	-0.8538	-0.0394	-0.8538	-0.0394	-0.8538
		8	-0.1001	-0.8582	-0.1001	-0.8582	-0.1001	-0.8582
	5	0.5	-0.2396	-0.9347	-0.2396	-0.9347	-0.2396	-0.9347
		0.8	-0.2827	-0.9132	-0.2827	-0.9132	-0.2827	-0.9132
		5	-0.3461	-0.8721	-0.3461	-0.8721	-0.3461	-0.8721
		8	-0.3507	-0.8687	-0.3507	-0.8687	-0.3507	-0.8687
	8	0.5	-0.3333	-0.8971	-0.3333	-0.8971	-0.3333	-0.8971
		0.8	-0.3541	-0.8816	-0.3541	-0.8816	-0.3541	-0.8816
		5	-0.3837	-0.8574	-0.3837	-0.8574	-0.3837	-0.8574
		8	-0.3858	-0.8556	-0.3858	-0.8556	-0.3858	-0.8556
5	0.5	0.5	1.9758	2.7099	1.9758	2.7099	1.9758	2.7099

	0.8	1.3206	1.08311	1.3206	1.08311	1.3206	1.08311
	5	-0.4770	-0.9557	-0.4770	-0.9557	-0.4770	-0.9557
	8	-0.6179	-0.8201	-0.6179	-0.8201	-0.6179	-0.8201
0.8	0.5	1.5949	1.7345	1.5949	1.7345	1.5949	1.7345
	0.8	0.8203	0.0853	0.8203	0.0853	0.8203	0.0853
	5	-0.5867	-0.8213	-0.5867	-0.8213	-0.5867	-0.8213
	8	-0.6843	-0.7056	-0.6843	-0.7056	-0.6843	-0.7056
5	0.5	0.2204	-0.6939	0.2204	-0.6939	0.2204	-0.6939
	0.8	-0.1743	-0.6876	-0.1743	-0.6876	-0.1743	-0.6876
	5	-0.6454	-0.6876	-0.6454	-0.6876	-0.6454	-0.6876
	8	-0.6735	-0.6565	-0.6735	-0.6565	-0.6735	-0.6565
8	0.5	0.05313	-0.3008	0.05313	-0.3008	0.05313	-0.3008
	0.8	-0.3109	-0.4021	-0.3109	-0.4021	-0.3109	-0.4021
	5	0.7379	-0.1575	0.7379	-0.1575	0.7379	-0.1575
	8	-0.7636	-0.1295	-0.7636	-0.1295	-0.7636	-0.1295

7.1 CONCLUSION

We defined and derived the four parameter Exponentiated Generalized Exponentiated Exponential distribution using Exponentiated Generalized Family as the generator and exponentiated exponential as base line distribution. Plot of EGEE density function for different parameter values are given in figure1. The graph shows that EGEE distribution can be monotonically decreasing (reversed J shape), left skewed, right skewed and unimodal depending on the shape parameters α, βbeta and θ.The cumulative distribution function (cdf) of EGEE in Figure 4 shows a satisfactory level of cdf not exceeding 1 on the y-axis. The shape parameter values have a strong influence on the shape of the graphs. As the parameter values reduce the movement on the x-axis tends to disappear. The hazard function graphical displayed in figure 7, shows a decreasing and constant display for parameter values less than zero, while the combination of other parameter values shows similar movements but at different pace on the graph. In Table 1, it is observed that the mean, standard deviation and mean deviation are increasing functions of the scale parameters θ when the other parameters are held constant. Increasing the scale parameter θ increases the mean, standard deviation and median deviation for fixed $\alpha, \beta and\ k$. The mean, standard deviation and median are increasing function of the shape parameters $\alpha, \beta and\ k$. An increase in one of the shape parameters when others are held constant increases the values of the mean, standard deviation and mean deviation of EGEE distribution. In Table 2, Skewness and kurtosis remain constant as the scale parameter θ increases when the shape parameters are held constant. An increase in the shape parameters reduces the values of skewness and kurtosis when the scale parameter is held constant. The EGEE distribution can be as an alternative distribution where the sub-models are applied.

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