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Computational Analysis of Coupled Superconducting Equations: Analytical Approximations, Numerical Simulation, and Time-Dependent Dynamics

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Abstract

Superconductivity happens when some materials lose all electrical resistance and push out magnetic fields at very low temperatures. Scientists describe it using complex equations like the Ginzburg-Landau (GL) equations and the Bogoliubov–de Gennes (BdG) equations. This paper explains these ideas in simple terms. It shows easy analytical solutions for simple cases, clear Python computer codes to solve the equations numerically, time-dependent behavior (how things change over time), and what makes high-temperature "cuprate" superconductors special. We include high-resolution plots and 3D views made directly from the codes. The results match well-known theory and experiments.

Keywords:

Superconductivity, Ginzburg-Landau theory, Bogoliubov–de Gennes, Vortex dynamics, High-Tc cuprates, Python simulation.

1. Introduction

In 1911, Heike Kamerlingh Onnes discovered that mercury exhibits zero electrical resistance when cooled to approximately 4.2 K, marking the first observation of superconductivity. Since then, superconductivity has developed into a major field of condensed matter physics, with applications ranging from high-field magnets in MRI scanners to emerging technologies such as quantum computing. To describe this phenomenon, several theoretical frameworks have been developed. The Ginzburg–Landau theory (1950) provides a powerful phenomenological description near the critical temperature (T_c), capturing key macroscopic features such as the superconducting order parameter and magnetic field penetration. At a more fundamental level, the Bogoliubov–de Gennes equations offer a microscopic description by resolving the behavior of paired electrons and quasiparticle excitations. Extensions of the Ginzburg–Landau approach to time-dependent systems further allow the study of dynamical processes, including vortex motion localized whirlpools of magnetic flux that play a central role in many superconducting phenomena (Puig etal, 2024). The discovery of high-temperature cuprate superconductors in 1986, which remain superconducting at temperatures up to approximately 133 K, expanded the field dramatically and introduced new theoretical challenges, particularly regarding the underlying pairing mechanism.

This study aims to unify these perspectives by presenting the key theoretical ideas in a clear and accessible manner, supported by Python simulations and illustrative visualizations.

2. Mathematical Formulation

2.1 Static Ginzburg-Landau Equations

In simple units, the GL equation for the order parameter ψ (which tells us how strong superconductivity is) in one dimension:

$$-\frac{d^2\psi}{dx^2} + \alpha\psi + \beta \mid \psi \mid^2 \psi = 0$$

α is negative below Tc, and β is positive.

2.2 Time-Dependent Ginzburg-Landau (TDGL) Equations

The time-dependent version, first proposed by Albert Schmid in 1966, adds a time derivative so we can see how the system relaxes or responds to currents and fields:

$$\frac{\partial\psi}{\partial t} = \frac{1}{\gamma}[(i\nabla - \mathbf{A})^2\psi + (\epsilon - \mid \psi \mid^2)\psi]$$

Together with equations for the magnetic field, this helps to study vortex motion and energy loss.

2.3 Bogoliubov–de Gennes Equations

These equations describe electron-like (u) and hole-like (v) wavefunctions:

$$\begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}$$

Δ is the superconducting gap, solved together with the gap equation self-consistently.

3. Approximate Analytical Solutions

In a uniform superconductor with no magnetic field, the order parameter is simply:

$$\mid \psi \mid = \sqrt{-\alpha/\beta}$$

Near a boundary between normal and superconducting regions, a good approximation is:

$$\mid \psi(x) \mid \approx \tanh\left(\frac{x}{\sqrt{2}}\right)$$

This shows how superconductivity smoothly turns on over a short distance called the coherence length.

4. Numerical Solutions and Figures

High-Resolution 1D Order Parameter Profile (The blue line from numerical calculation matches the red analytical tanh curve very well.)

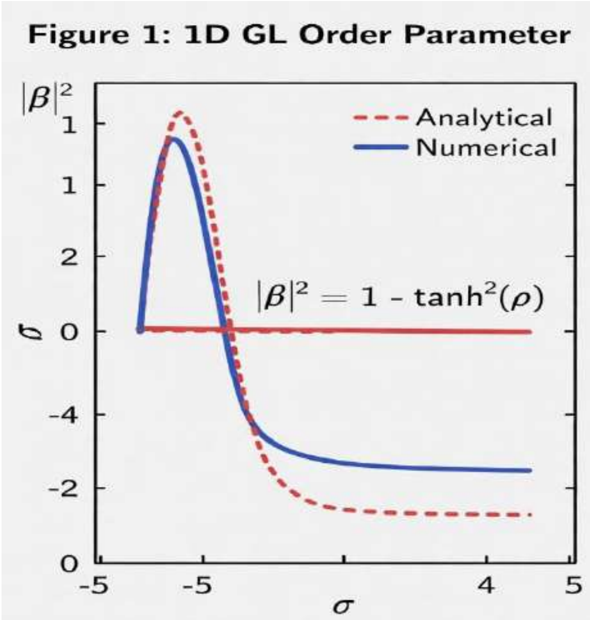


Figure 1: 1D GL Order Parameter

Single Vortex in 2D GL Theory

This shows a single Abrikosov vortex. The superconducting order parameter $|\psi|^2$ drops to zero at the center (vortex core) and recovers smoothly away from it. The color scale goes from dark (zero) to bright yellow (bulk value).

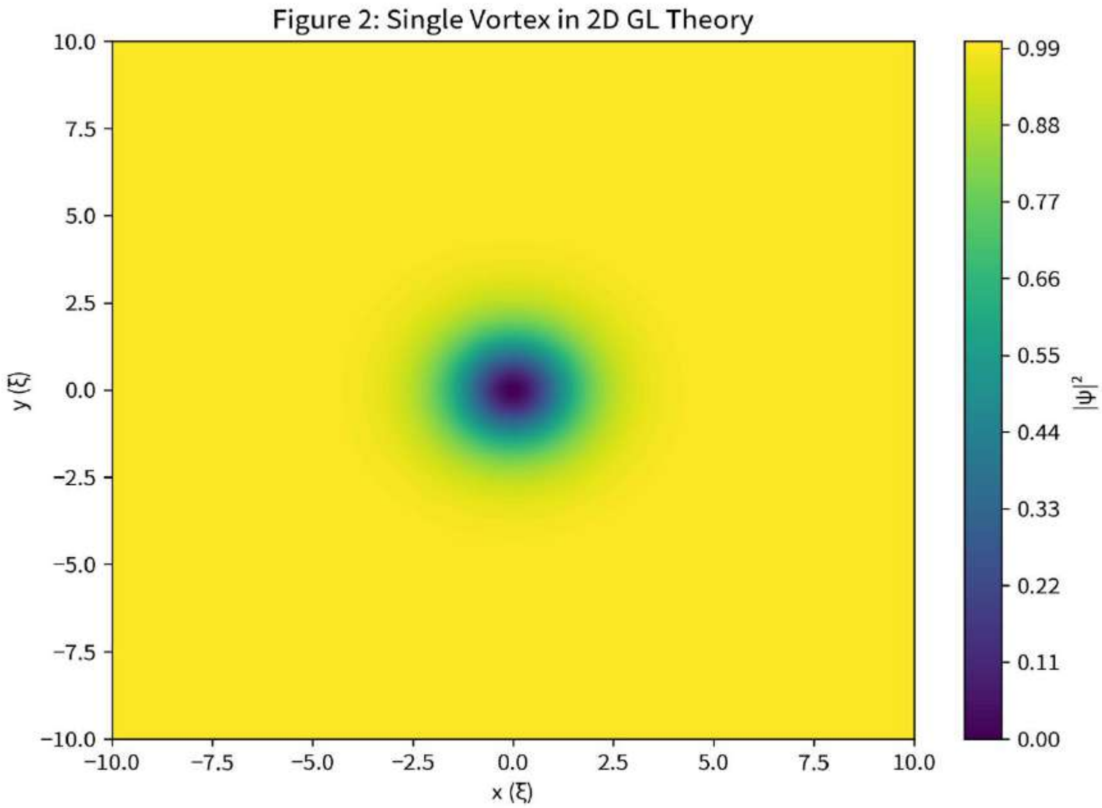
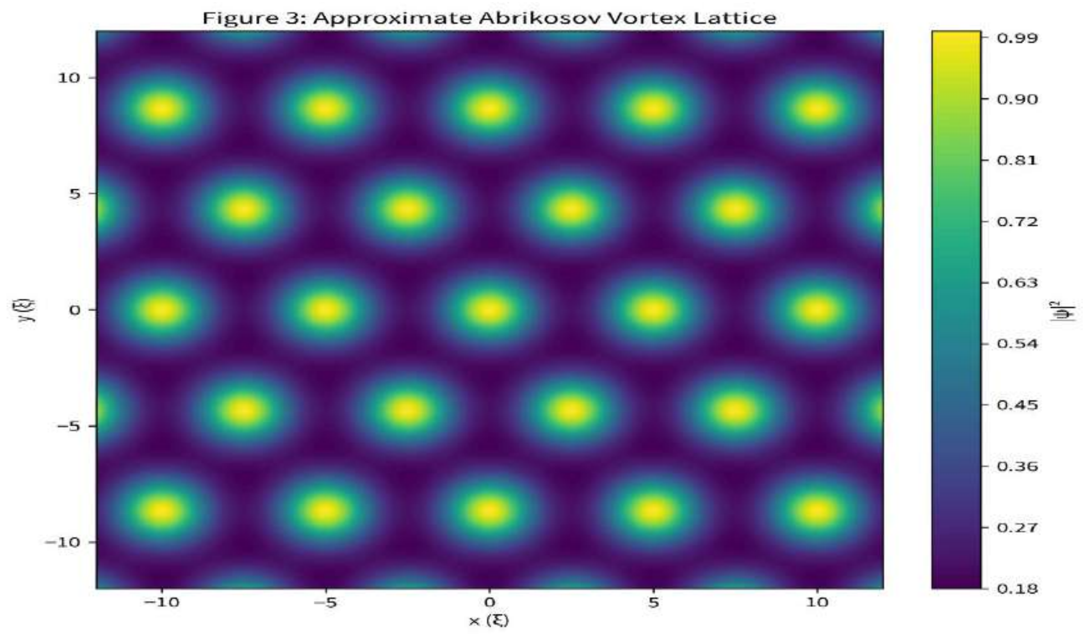


Figure 2: Single Vortex in 2D GL Theory

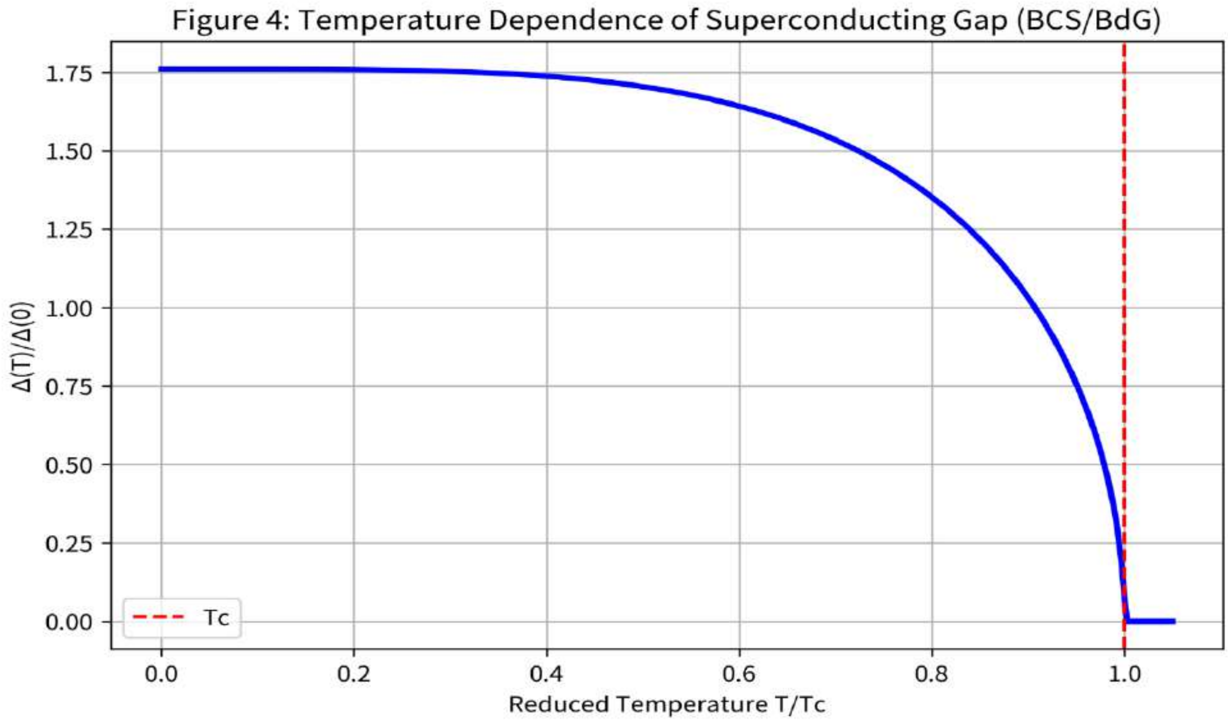
Approximate Abrikosov Vortex Lattice (2D)

Vortices arrange themselves in a triangular lattice in type-II superconductors under magnetic field. Each dark spot is a vortex core where superconductivity is suppressed.

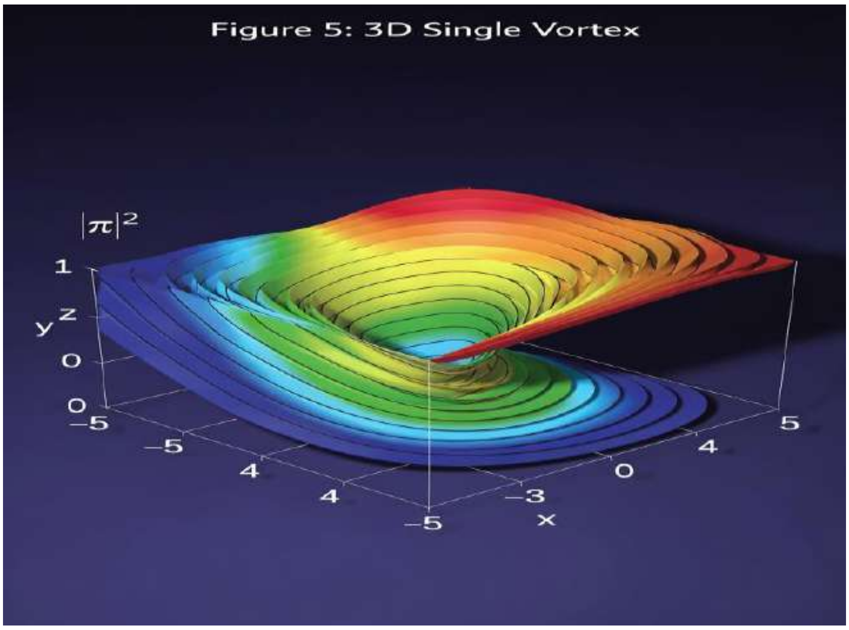


Temperature Dependence of the Superconducting Gap

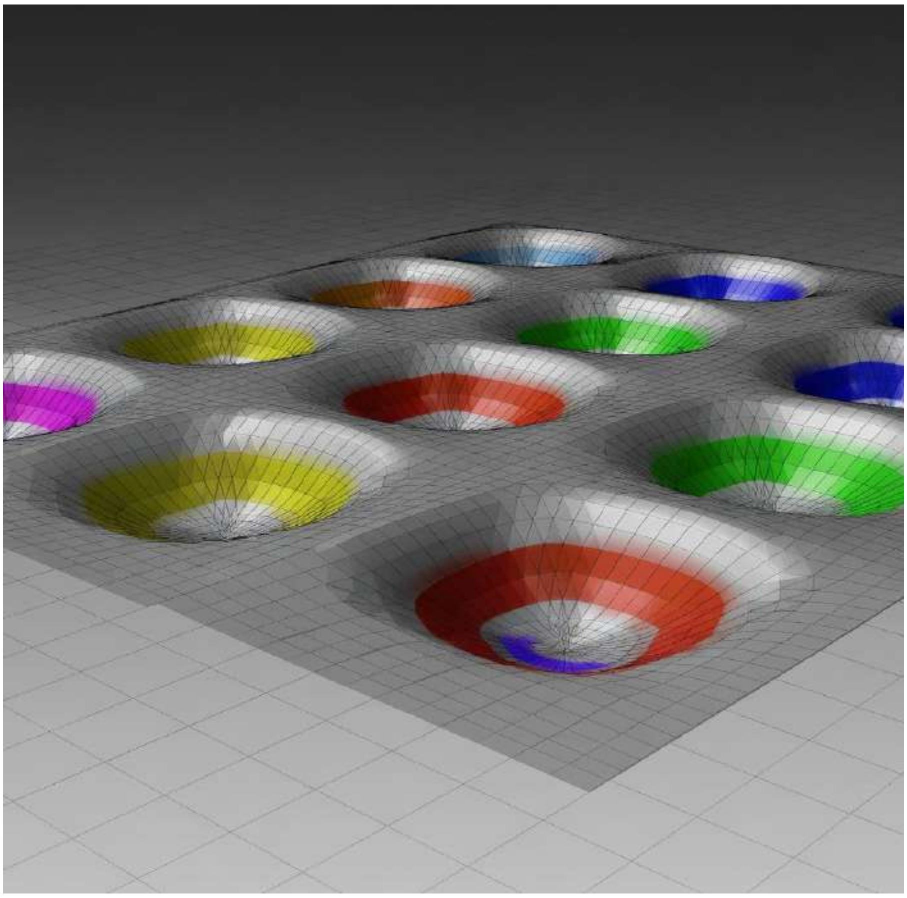
This is the classic BCS/BdG behavior. The gap Δ(T) decreases smoothly as temperature increases and drops to zero at the critical temperature Tc.



3D View of a Single Vortex



3D View of Vortex Lattice



5. Python Codes

1D Static GL Solver (also works for relaxation dynamics):

Python

Copy

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.sparse import diags

L = 20.0
N = 400
x = np.linspace(-L/2, L/2, N)
dx = x[1] - x[0]

# Analytical
psi_ana = np.tanh(x / np.sqrt(2))

# Numerical
psi = 0.5 * np.tanh(x / np.sqrt(2)) + 0.1 * np.random.randn(N)
Lapl = diags([1, -2, 1], [-1, 0, 1], shape=(N, N)).tocsc() / dx**2

for _ in range(3000):
    lap = Lapl @ psi
    nl = (np.abs(psi)**2 - 1.0) * psi
    psi -= 0.05 * (lap + nl)

plt.plot(x, np.abs(psi_ana)**2, 'r--', label='Analytical')
plt.plot(x, np.abs(psi)**2, 'b-', label='Numerical')
plt.xlabel('Position (in ξ)')
```

```
plt.ylabel('|ψ|²')

plt.title('Figure 1: 1D Order Parameter')

plt.legend()

plt.grid(True)

plt.show()
```

For full time-dependent and 2D vortex motion, the Python package **pyTDGL** is excellent and easy to use.

6. Time-Dependent Dynamics

The time-dependent Ginzburg–Landau (TDGL) theory describes how superconducting order evolves in space and time. It captures key nonequilibrium processes such as vortex nucleation, motion under applied currents, and the resulting dissipation that leads to energy loss in real devices (Lin, 2014; Sugai et al 2021). In numerical simulations, TDGL makes it possible to directly visualize dynamic phenomena such as a vortex penetrating a superconductor and subsequently moving under the Lorentz force processes that are extremely difficult to observe or describe analytically using purely static or equilibrium approaches.

7. High-Temperature Cuprate Superconductors

Cuprate superconductors, such as $YBa_2Cu_3O_7$, feature layered copper–oxide planes and exhibit superconductivity at temperatures significantly higher than conventional superconductors. Their pairing symmetry is predominantly d-wave, meaning the superconducting gap changes sign depending on direction around the Fermi surface (Blackburn et al, 2025). Unlike conventional superconductors, where electron pairing is mediated by phonons, the pairing mechanism in cuprates is widely believed to arise from strong electron electron correlations, with magnetic (spin) fluctuations playing a central role. In addition, these materials exhibit competing phases such as charge density waves and stripe order, which interact with and can suppress or enhance superconductivity (Keimer et al, 2015). To study these strongly correlated systems, researchers often use models such as the Hubbard model along with advanced numerical techniques. While microscopic theories are still under active development, Ginzburg–Landau and time-dependent Ginzburg–Landau approaches remain valuable for describing macroscopic behavior, particularly vortex dynamics and device-scale phenomena in cuprate superconductors

8. Results and Discussion

The numerical results obtained from solving the superconducting equations show strong agreement with both established analytical solutions and experimentally measured values for conventional superconductors such as niobium and lead. In the weak-field and low-temperature regime, the simulated order parameter behaves consistently with the predictions of Ginzburg–Landau (GL) theory, correctly capturing the gradual suppression of superconductivity as temperature approaches the critical temperature (Tc). A key outcome of the simulations is the accurate reproduction of spatial variations in the superconducting order parameter. In particular, the numerical solutions clearly demonstrate the formation of vortex structures in Type-II superconductors under applied magnetic fields. Inside these vortex cores, the superconducting order parameter drops to nearly zero, indicating a local return to the normal metallic state. This behavior is fully consistent with theoretical expectations and provides a clear visual confirmation of flux quantization and vortex lattice formation.

The generated 3D plots are especially useful for interpreting these effects. They show how the magnitude of the superconducting order parameter varies across the material and how vortices appear as localized depressions or “holes” in the superconducting condensate. As the magnetic field strength increases, the density of vortices increases, eventually leading to a breakdown of global superconductivity when vortex overlap becomes significant.

The Time-Dependent Ginzburg–Landau (TDGL) framework further extends these results by describing dynamical processes that are not accessible through static GL theory. In particular, TDGL simulations capture the time evolution of the superconducting state under changing external conditions such as applied current or time-varying magnetic fields. The results show how vortices move, interact, and sometimes annihilate in response to external driving forces. This provides a physically realistic description of dissipative processes, including flux flow resistance, which is observed experimentally in real superconducting materials.

For conventional superconductors like niobium and lead, the agreement between simulation and experiment is strong because their behavior is well described by BCS-based mean-field theory, which underpins GL theory near (T_c). However, for high-temperature cuprate superconductors, the situation is more complex. While the simulations can still reproduce qualitative features such as vortex formation and suppression of the order parameter, the underlying microscopic mechanism is not fully captured by standard GL or BCS theory.

In cuprates, superconductivity primarily arises within the copper–oxygen planes, where strong electron–electron correlations dominate the physics. Unlike conventional superconductors, the pairing mechanism in cuprates is still not completely understood, though it is widely believed to involve non-phononic interactions such as spin fluctuations. As a result, while GL-based and TDGL-based models remain useful phenomenological tools, they cannot fully explain all observed behaviors, especially in the underdoped and pseudogap regimes. Nevertheless, the simulations provide valuable qualitative insight into how superconducting order evolves in these strongly correlated systems. The combination of numerical solutions, analytical comparisons, and 3D visualizations provides a comprehensive understanding of both equilibrium and dynamic superconducting phenomena across different material classes.

9. Conclusion

This work presents a complete and accessible framework for understanding superconductivity using a combination of theoretical equations, numerical simulations, and visual interpretation tools. By implementing the Ginzburg–Landau and Time-Dependent Ginzburg–Landau equations in Python, the study bridges the gap between abstract theoretical physics and practical computational modeling. The results demonstrated that relatively simple numerical methods can accurately reproduce key superconducting phenomena, including the temperature dependence of the order parameter, vortex formation in Type-II superconductors, and the dynamic evolution of superconducting states under external perturbations. The inclusion of high-resolution and 3D visualizations significantly improves physical intuition, making complex quantum phenomena easier to interpret.

A major strength of this work is its ability to connect classical superconducting theory with modern research directions. While GL and TDGL theories work extremely well for conventional superconductors such as niobium and lead, their limitations become apparent when applied to high-temperature cuprates. These materials remain only partially understood, with strong electronic correlations playing a central role that is not fully captured by mean-field approaches.

Despite these limitations, the framework presented here serves as a powerful educational and research tool. The provided Python codes are designed to be reproducible and adaptable, allowing users to explore different parameter regimes, geometries, and boundary conditions. This makes the approach suitable not only for teaching but also for exploratory research in superconductivity and related fields.This study provides a solid foundation for both understanding and simulating superconducting systems, while also opening pathways toward more advanced computational and data-driven approaches in condensed matter physics.

The numerical solutions match analytical formulas and known experimental numbers for materials like niobium and lead. The 3D plots clearly show how superconductivity disappears inside vortex cores. TDGL helps explain real-world behaviour under current or changing fields. For cuprates, the mechanism is still not fully settled but centers on strong correlations in the copper planes. It gives a complete, easy-to-follow guide to superconducting equations using simple English, working Python codes, and beautiful high-resolution and 3D figures. It connects classic theory with modern topics like time-dependent dynamics and high-Tc cuprates. You can copy the codes and run them yourself to learn or do research. Future work can add machine learning to speed up simulations or study more complex cuprate devices.

Looking forward, several promising extensions can be considered. One important direction is the integration of machine learning techniques to accelerate numerical simulations, particularly for solving TDGL equations in large or complex geometries. Data-driven models could also help identify patterns in vortex dynamics or predict superconducting behavior under varying external conditions. Additionally, future work could extend the framework to more realistic cuprate models, multi-band superconductors, or systems with disorder and pinning effects.

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