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EXISTENCE, STABILITY AND BOUNDEDNESS OF PERIODIC SOLUTIONS FOR CERTAIN NONLINEAR BOUNDARY VALUE PROBLEM OF FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

Ezugorie I. G.
Department of Mathematics, Enugu State University of Science and Technology, Enugu State.
Email: ikegodezugorie@esut.edu.ng

ABSTRACT:

In this paper, some new conditions on the existence, stability and boundedness of periodic solutions for certain nonlinear boundary value problems were investigated. Construction of a complete Lyapunov function for higher order differential equations were not all that easy and therefore a new method for investigating and proving the existence, stability and boundedness was considered. The method of Leray-Schauder fixed point theorem provided existence of periodic solutions which depended on the availability of suitable boundedness results. In some cases, boundedness results were very difficult to establish due to the nature of the Lyapunov function involved and the method of the integrated equation was used as a mode for estimating apriori bounds for the fourth order differential equation. The aim of using integrated equation was to ameliorate the technical problems arising from the construction of Lyapunov function for higher order differential equations which was considered to be cumbersome and complex. However, our results generalize and complement some existing results in literature.

Keywords:

Existence, Stability, Leray-Schauder fixed point theorem, integrated equation, Periodic solution.

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1. Introduction

Consider the fourth order nonlinear differential equation of the form

$$x^4 + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + h(x) = p(t) \qquad (1.1)$$

which had its origin following investigations Ezeilo (1963) did on the third order differential equation. Harrow (1967) extended the ideas to the behavior of the system of a fourth order system

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = u$$

$$\dot{u} = -au - bz - cy - h(x) + p(t) \qquad (1.2)$$

The existence of periodic solutions to more general form of (1.1) have been studied by Tejumola (1969), Ezeilo and Onyia (1983) and Ressig (1974). Recently on the non-resonant oscillations for some fourth order differential equation. Ezeilo and Onyia (1984) considered the following equation

$$x^4 + a_1x^3 + a_2\ddot{x} + h(x)\dot{x} + a_4x = p(t) \qquad (1.3)$$

$$x^4 + a_1x^3 + a_2\ddot{x} + g(\dot{x}) + a_4x = p(t) \qquad (1.4)$$

For a.e $t \in [0,2\pi]$ which is subjected to 2π boundary conditions on $[0,2\pi]$ where a_1, a_2, a_4 are constants and $g: \mathbb{R} \to \mathbb{R}$ are continuous functions. $p: \mathbb{R} \to \mathbb{R}$ is also continuous. Solutions here are caratheodory sense.

Over the years, several theories have emerged on the methods of finding the existence of periodic solutions. For instance Mawhm and Wand (1982) and Bockelman (2006) have used the degree theory. On the use of lyapunov functions see Ezeilo (1962,1963, 1966),Tejumola (1969), Ezeilo and Onyia (1984) and Ezeilo and Nkashama (1985). On the use of frequency domain approach see Afuwape (1981). Most recently, Eze et al (2013), Eze and Aja (2015), Eze et al (2017, 2018, 2019a, 2019b, 2019c, 2019d, 2021), Eze and Udaya (2020), Osu et al (2020), have used implicit function theorem as well as the lyapunov method to prove the existence of periodic solutions. Guoshan and Piezhao (2018) and Bernstein and Bhat (1994) used lyapunov method while Kearfott (2008) used the interval Newton-method.

Motivated by the above results and ongoing research in this direction, the purpose of this paper is to prove existence and bounds of periodic solutions for certain nonlinear differential equation of the fourth order using Leray-Schauder fixed point technique and the integrated equation as the mode for estimating the apriori bounds for higher order differential equation. The aim of using the integrated equations is to tackle technical problems arising from the construction of lyapunov function using the method adopted from Cartwright (1956). However, our results generalize and complement some existing results in literature.

2. Preliminaries

Theorem 2.1 Ezeilo (1999a). Let $a_1 \neq 0$ and $a_4 \neq 0$ and suppose that the function $H(x)$ defined by $H(x) = \int_0^x h(s)ds$ satisfies $m^2 + \theta_1(|x|) < a^{-1}x^{-1}H(x) < (m+1)^2 = \theta_2(|x|, |x|) > r$ where m is an integer and $\theta_i: [0, +\infty] \to \infty$ $(i = 1,2)$ are two functions such that $|x|\theta_i(|x|) \to \infty$ as $|x| \to \infty$ $(i = 1,2)$. Then equation (1.3) with the boundary condition on $[0,2\pi]$ has at least solution for every $p \in L^1[0,2\pi]$ and for arbitrary a_2.

Theorem 2.2 Ezeilo (1999b) Let $a_1 \neq 0$ and $a_4 \neq 0$ and suppose that g satisfies $m^2 + \eta_1(|y|) \leq a^{-1}y^{-1}g(x) \leq (m+1)^2, |x| > r$ where $m > 0$ is an integer. $\eta_i: [0, \infty] \to +\infty$ $(i = 1,2)$ are two functions such that $|y|\eta_i(|y|) \to \infty$ as $|y| \to \infty$ $(i = 1,2)$. Then the $2\pi -$ periodic boundary value problem (1.4) has at least one solution for every $p \in L^1[0,2\pi]$ and for arbitrary a_2. Ezeilo and Tejumola (2001) considered the fourth order differential equation

$$x^{(4)} + \varphi(\dddot{x}) + \varphi(\ddot{x}) + \varphi(\dot{x}) + f(t, x_1, x_2, x_3, x_4) = p(t, x_1, x_2, x_3, x_4) \qquad (1.5)$$

in which $\varphi, \theta: \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R}^4 \to \mathbb{R}$, $p: [0,2\pi] \times \mathbb{R}^4 \to \mathbb{R}$ are C^0 and $\varphi: \mathbb{R} \to \mathbb{R}$ is C^1 and assume that p has a periodic 2π in t uniformly with respect to other variables and that

$$|p(t, x_1, x_2, x_3, x_4)| \leq A^* < \infty \qquad (1.6)$$

for some constants A^* for all values of t, x_1, x_2, x_3, x_4. Then the result follows.

Theorem 2.3 Ezeilo and Tejumola (2001). Suppose further to the basic assumption in theorem 2.2 that there exist a constant $a_2 > 0$ such that

$$|f(x_1, x_2, x_3, x_4)| \leq a_2 \text{ for all } x_1, x_2, x_3, x_4 \qquad (1.7)$$

$$\inf_{|x| \leq 1} \varphi'(x) > \frac{1}{4}\alpha_2^2 \qquad (1.8)$$

$$\max_{x_2^2 + x_2^2 \leq A^2} |f(x_1, x_2, x_3, x_4)| \leq kA \qquad (1.9)$$

For arbitrary x_1, x_4. Then equation (1.5) subject to the boundary conditions

$$D^r x(0) = D^r x(2\pi) \ \ r = 0,1,2,3 \ , D = \frac{d}{dt} \qquad (1.10)$$

has at least one solution for arbitrary φ and θ.

Definition 2.3 A solution $x(t)$ of a differential equation $\dot{x} = f(t, x), t, x \in \mathbb{R}$ is said to be apriori bounded if there are obvious causes for $x(t)$ to be bounded. These reasons emanate from the space $\mathbb{R}^n$ and any transformation T defined on the space. Thus $|x|_\infty \leq A_0$. Then $x(t)$ is apriori bounded.

Lemma 2.4 (Schaefer's Lemma) Let T be a compact transformation of a normed linear space S into itself. Let $\lambda \in (0,1)$ then either there is $X \in S$ such that

$$X = \lambda TX \tag{1.11}$$

Or the set $\{X/X \in S: X = \lambda_1 TX_1 \; 0 < \lambda_1 < 1\}$ is unbounded. Clearly, in order to be able to conclude that (1.11) has a solution $X \in S$ for each $\lambda \in (0,1)$, it will be enough to verify the existence of a constant A_0 $(0 < A_0 < \infty)$ independent of $\lambda \in (0,1)$ such that for every $X \in S$ satisfying (1.11), the relation $\|x\| \leq 0$ hold where $\|x\|$ denotes the norm of $X \in S$.

Definition 2.5 (Integrated Equation). This is an equation obtained mainly from pre-multiplication of a given differential equation by $x, \dot{x}, \ddot{x}$ etc. as the case might be and thereafter integrating with respect to t from $t = 0$ to $t = T$. The resulting equation is referred to as 'integrated equation'. The equation is similar to a lyapunov function which is applicable in determining stability, instability, boundedness, ultimate boundedness and periodicity of solutions in ordinary differential equations. However, it is difficult to construct a suitable lyapunov function for higher order nonlinear differential equation. Even when such a lyapunov function is constructed, it might not be utilized in estimating $\int_0^{2\pi} \ddot{x}^2 \, dx$ or $\int_0^{2\pi} \dot{x}^2 \, dx$ for a possible solution $x(t)$ of associated parameter differential equation. For instance, the method adopted by Cartwright (1956) being extended to a fourth order differential equation

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = 0 \tag{1.12}$$

and its nonlinear forms has a lot of difficulties which are shown below. The procedure is to transform the equation (1.12) into system form

$$\dot{x} = y, \dot{y} = z, \dot{z} = w, \dot{w} = -a_1 w - a_2 z - a_3 y - a_4 x \tag{1.13}$$

or writing compactly

$$\dot{X} = AX \tag{1.14}$$

$$\text{Where } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \tag{1.15}$$

The method which is being discussed is based on the fact that the matrix A defined in (1.15) has all its eigenvalues with negative real parts. Then from the general theory that correspond to any positive definite quadratic form u, there exist another positive quadratic form V such that

$$\dot{V} = -u \tag{1.16}$$

We choose the most general quadratic form of order two and pick the coefficients in the quadratic form to satisfy equation (1.16) along the solution paths of equation (1.13). Let V be defined by

$$2V = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2 + \alpha_4 w^2 + 2\alpha_5 xy + 2\alpha_6 wz + 2\alpha_7 xw$$

$$+2\alpha_8 yz + 2\alpha_9 yw + 2\alpha_{10} zw \qquad (1.17)$$

At this stage, there are already too many coefficients to context with. Now

$$\dot{V} = \alpha_1 xy + \alpha_2 yz + \alpha_3 zw - a_1\alpha_4 w^2 - a_2\alpha_4 zw - a_3\alpha_4 yw - a_4\alpha_4 xw + \alpha_4 xz$$

$$+\alpha_5 y^2 + \alpha_6 yz + \alpha_6 xw - a_1\alpha_7 xw - a_2\alpha_7 xz - a_3\alpha_7 xy - a_4\alpha_7 x^2 + \alpha_8 z^2 + \alpha_8 yw + \alpha_9 zw$$

$$-a_1\alpha_9 yw - a_2\alpha_9 y^2 - a_3\alpha_9 y^2 - a_4\alpha_9 zw + \alpha_{10} w^2 - \alpha_{10} zw$$

$$-a_2\alpha_{10} z^2 - a_3\alpha_{10} yz - a_4\alpha_{10} xz \qquad (1.18)$$

Equation (1.18) is very boring and cumbersome for consideration in terms of coefficients

Table 1.1 showing terms and coefficients of (1.18)

Terms	Coefficients
x^2	$-a_4\alpha_7$
y^2	$\alpha_5 - a_3\alpha_7$
z^2	$\alpha_8 - a_3\alpha_9$
w^2	$\alpha_{10} - a_1\alpha_4$
xy	$\alpha_1 - a_3\alpha_7 - a_4\alpha_9$
xz	$-a_4\alpha_{10}$
xw	$\alpha_6 - a_4\alpha_4 - a_1\alpha_7 - a_3\alpha_{10}$
yz	$\alpha_2 + \alpha_6 + \alpha_7 - a_2\alpha_9 - a_3\alpha_{10}$
yw	$\alpha_8 - a_3\alpha_4 - a_1\alpha_9$
zw	$a_3 - a_2\alpha_4 + \alpha_9 - a_1\alpha_{10}$

Equation (1.17) and (1.18) have given us an insight to the difficulties involved in the above method. Therefore to avoid these difficulties arising from the complexities in $\dot{V}$ through the solution paths of the above table, we opt for the method of integration.

Also, the nonlinear fourth order parameter λ −dependent equation $(0 \le \lambda \le 1)$

$$x^{(4)} + \lambda\varphi(\dot{x})\ddot{x} + h_\lambda(x, \dot{x}, \ddot{x}, \dddot{x})\ddot{x} + \lambda\theta(\dot{x}) + f_\lambda(x) = 0 \qquad (1.19)$$

Where $h_\lambda(x, \dot{x}, \ddot{x}, \dddot{x})\ddot{x} = (1-\lambda)\alpha_2\ddot{x} + h_\lambda(x, \dot{x}, \ddot{x}, \dddot{x})\ddot{x}$ and $f_\lambda(x) = (1-\lambda)\alpha_4 x + \lambda f(x)$

In system form equation (1.19) can be written as

$$\dot{x} = y, \dot{y} = z, \dot{z} = w, \dot{w} = -\lambda\varphi w - h_\lambda z - \lambda\theta(y) - f_\lambda(x) \qquad (1.20)$$

The main objective here is to construct an integrated equation for the system (1.20) through the constant coefficient fourth order differential equation of (1.12) or the equivalent fourth order system of (1.13). Multiplying (1.12) by $\ddot{x}$ and integrating with respect to t from $t = 0$ to $t = T$, we have

$$[uz + \tfrac{1}{2}a_1\ddot{x}^2 + \tfrac{1}{2}a_3\dot{x}^2 + a_4 x\dot{x}]_0^T - \int_0^T \dddot{x}^2 ds + \int_0^T a_2\,\ddot{x}^2 ds - \int_0^T a_4\dot{x}^2 ds = 0 \qquad (1.21)$$

Now replacing $[uz + \frac{1}{2}a_1\ddot{x}^2 + \frac{1}{2}a_3\dot{x}^2 + a_4x\dot{x}]_0^T$ by $V(x, y, z, w)$ in (1.21) where (x, y, z, w) are all functios of t, we have

$$V - \int_0^T \dddot{x}^2 ds + \int_0^T a_2\,\ddot{x}^2 ds - \int_0^T a_4\dot{x}^2 ds = 0 \tag{1.22}$$

Which implies that

$$V = \int_0^T \dddot{x}^2 ds - \int_0^T a_2\,\ddot{x}^2 ds + \int_0^T a_4\dot{x}^2 ds \tag{1.23}$$

But from (1.24)

$$\dot{V} = \dddot{x}^2 - a_2\ddot{x}^2 + a_4\dot{x}^2 \tag{1.24}$$

If we put $u = -\dddot{x}^2 + a_2\ddot{x}^2 - a_4\dot{x}^2$ in (1.24) then (1.24) becomes (1.21) which is a confirmation that the present paper is in line with the theory on the construction of lyapunov function. Therefore our V for the system (1.13) could be given by

$$V = uz + \frac{1}{2}a_1\ddot{x}^2 + \frac{1}{2}a_3\dot{x}^2 + a_4x\dot{x} \tag{1.25}$$

Next, we find a V as corresponding to a V in (1.25) corresponding to the nonlinear system (1.20). Without the loss of generality, the comparison equations (1.20) and (1.13) indicate that (1.20) is equivalent to (1.13) if

$$\left.\begin{array}{c} \lambda\varphi(z)\text{is replaced by } a_1 \\ \lambda\theta(y)\text{is replaced by } a_2 y \\ f_\lambda \text{ is replaced by } a_4 x \end{array}\right\} \tag{1.26}$$

We observed that a_2 does not appear in V as given in (1.25) so the correlation (1.26) suggests that our V for the nonlinear system (1.20) could be given by

$$V = v(x, y, z, w) = \lambda\int_0^x s\varphi(s)ds + uz + yf_\lambda(x) + \lambda\int_0^y \theta(s)ds \tag{1.27}$$

On the applicability of V so far constructed for nonlinear fourth order differential equation, we note that the equation have been in use from the time of Cartwright (1956) for second order differential equations. It continued in the papers by Ressig (1972), Voitovich (2011) and Tiryaki .and Tunc (1995). However, the fact remain that there is no available literature on how thses functions could be obtained. In this paper, there is an improvement on the integrated equations as it concerns the application of Leray-Schauder fixed point technique to fourth order differential equation.

4. Results

Consider the differential equation

$$x^{(4)} + f(\dddot{x}) + g(\ddot{x}) + h(\dot{x}) + a_4x = p(t, \dot{x}, \ddot{x}, \dddot{x}) \tag{1.28}$$

With boundary conditions

$$D^{(r)}x(0) = D^{(r)}x(2\pi);\ r = 0,1,2,3.\ D = \frac{d}{dt} \tag{1.29}$$

Where a_4 is a constant greater than zero and $f = f(\ddot{x}), g = g(\dot{x}), h = h(\dot{x})$ and p is continuous functions depending on their arguments with p being 2π periodic in t. Equation (1.28) is a more general form of equation (1.1) where b_1, b_2, b_3 are all not constants. It could also be seen as one of the configuration of the equation

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \quad (1.30)$$

in which a_1, a_2 and a_3 are not constants. Our consideration will be based on auxiliary equation

$$r^4 + a_1r^3 + a_2r^2 + a_3r + a_4 = 0 \quad (1.31)$$

of (1.30) for $p \equiv 0$, which has a root of the form $r = i\beta$ (β an integer) if the two equations

$$\beta^4 - a_2\beta^2 + a_4 = 0 \text{ and } i\beta(a_3 - a_1\beta^2) = 0 \quad (1.32)$$

Are satisfied simultaneously. Thus the corresponding non-homogenous equation (1.30) together with boundary conditions (1.29) have no non-trivial solutions if either

$$X(\beta) = \beta^4 - \alpha_2\beta^2 + a_4 \neq 0 \quad (1.33)$$

$$a_3 - a_1\beta^2 \neq 0 \quad (1.34)$$

From (1.33) we obtain

$$X(\beta) = (\beta^2 - \tfrac{1}{2}a^2)^2 + a_4 - \tfrac{1}{4}a_2^2 \neq 0 \quad (1.35)$$

By completing the squares from (1.35)

$$a_4 > \tfrac{1}{4}a_2^2 \quad (1.36)$$

follows. This in turn implies that equation (1.30) subject to conditions (1,29) have at least one $2\pi -$ periodic solution if p is bounded and $2\pi -$periodic in t for arbitrary a_1 and a_3. The equation (1.36) and its generalization to nonlinear terms have been used extensively by scholars in the proof of existence of $2\pi -$ periodic solutions for nonlinear fourth order differential equation. For more results on the configurations of equation (1.30), Tiryaki and Tune (1995) proved existence of periodic solutions for the equation

$$x^{(4)} + f_1(\ddot{x})\dddot{x} + f_2(\ddot{x})\ddot{x} + f_3(\dot{x}) + f_4(x) = 0$$

However, Tiryaki's result was for the rivial solution. Tejumola (2006) proved also existence of nontrivial solution for the equation

$$x^{(4)} + g_1(\dddot{x}) + g_2(\ddot{x}) + g_3(\dot{x}) + b_4(x) = p_2(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

Wth $p_2 \equiv 0$ and g_2 arbitrary..

Our objective here is to consider equation (1.34) among others as a condition for achieving existence of periodic solutions for equations (1.28) and (1.29). Note that equation (1.28) is comparable with equation (1.30) if

$$\left.\begin{array}{r} f(\dddot{x})\ is\ replaced\ by\ a_1\,\dddot{x} \\ g(\dot{x})\ is\ replaced\ by\ a_2\ddot{x} \\ h(\dot{x})\ is\ replace\ by\ a_3\dot{x} \end{array}\right\} \quad (1.37)$$

The functions $f(\dddot{x})$ and $h(\dot{x})$ replacing $a_1\dddot{x}$ and $a_3\dot{x}$ suggest that $\frac{f(\dddot{x})}{\dddot{x}}$ and $h'(\dot{x})$ are suitable replacement for a_1 and a_3 respectively. So in (1.34) suggests that an existence of a 2π −periodic solution might be provable for equation (1.28) where p is bounded and 2π −perodic in t for arbitrary any a and a_4. Since $y = \dot{x}, z = \ddot{x}, u = \dddot{x}$ then $\frac{f(\dddot{x})}{\dddot{x}} = \frac{f(u)}{u}$ and $h'(\dot{x}) = h'(y)$

(1.38)

Thus we have the following

Theorem 4.1 Suppose further to the basic assumptions on f, g, h, a_4 and p that

1. There are constants $a_1 > 0, a_3 > 0$ such that $\frac{f(u)}{u} \geq a_1,\ a_1 \neq 0$ (1.39)

2. The function $h(y)$ is such that $h'(y) \leq a_3 < a_1$ (1.40)

3. $a_1^{-1}a_3 \neq \beta^2$ (1.41)

4. The function p is bounded and 2π −periodic in t. Then equation (1.28) and (1.29) have at least one 2π −periodic solution for arbitrary a and a_4.

Note: The above theorem (4.1) is on a more general form of equation (1.3) and is as a result based on equation (1.34) which is rare in literature.

Theorem 4.2 In addition to the basic assumption on g, h, a_3, a_4 and p. Suppose that

$$a_4 > \frac{1}{4}h(x,\dot{x},\ddot{x},\dddot{x})V(x,\dot{x},\ddot{x},\dddot{x}) \quad (1.42)$$

The function p is bounded and 2π −periodic in t. Then equation (1.28) and (1.29) have at least one 2π −periodic solution for arbitrary g and a_3.

Proof of theorem 4.2

The proof of theorem 4.2 is by Leray-Shauder fixed point technique and instead of considering

$$x^4 + g(\dddot{x}) + h(x,\dot{x},\ddot{x},\dddot{x})\ddot{x} + a_3\dot{x} + a_4x = p(t,x,\dot{x},\ddot{x},\dddot{x}) \quad (1.43)$$

with boundary conditions

$$D^r x(0) = D^r x(2\pi),\ r = 0,1,2,3,\ D = \frac{d}{dt} \quad (1.44)$$

We consider the parameter λ − dependent equation

$$x^4 + \lambda g(\dddot{x}) + h_\lambda(x,\dot{x},\ddot{x},\dddot{x})\ddot{x} + a_3\dot{x} + a_4x = \lambda p \quad (1.45)$$

Where $h_\lambda(x,\dot{x},\ddot{x},\dddot{x})\ddot{x} = (1-\lambda)a_2\ddot{x} + \lambda h(x,\dot{x},\ddot{x},\dddot{x})\ddot{x}$

By setting $\dot{x} = y, \dot{y} = z, \dot{z} = u, \dot{u} = -\lambda g(u) - \lambda g(z) - a_3y - a_4x + \lambda p$, equation (1.45) can be written compactly in the matrix form

$$\dot{X} = AX + \lambda F(X,t) \tag{1.46}$$

$$\text{where } X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.47}$$

with $Q = p - g(u) - h(z) + a_2 z$. Equation (1.45) reduces to linear equation

$$x^4 + a_2\ddot{x} + a_3\dot{x} + a_4 x = 0 \tag{1.48}$$

And to equation (1.43) if $\lambda = 1$. The eigenvalues of matrix A defined by equation (1.47) are the roots of the auxiliary equation of (1.48) which have no roots of the form $r = i\beta$ (β is an integer). If

$$a_4 > \frac{1}{4}a_2^2 \tag{1.49}$$

The implication of (1.49) is that (1.48) to (1.46) have no nontrivial solutions. Therefore the matrix $(e^{-2\pi A} - I)$ (I being the identity 4×4 matrix) is invertible. Thus $X = X(t)$ is a 2π periodic solution of (1.46) if and only if

$$X = \lambda TX,\ 0 \le \lambda \le 1 \tag{1.50}$$

Where the transformation T is defined by

$$(TX)(t) = \int_0^{T+2\pi} (e^{-2\pi A} - T)^{-1} e^{A(t-s)}\, F\big(X(s)\big) ds \tag{1.51}$$

Let S be the space of all real-valued continuous and 4-vector function $X(t) = (x(t), y(t), z(t), u(t))$ which are of period 2π with norm

$$\|X\|_s = \underset{0 < t < 2\pi}{Sup} \{|x(t)| + |y(t)| + |z(t)| + |u(t)|\} \tag{1.52}$$

If the operator T defined by (1.51) is a compact mapping of S into itself, then it suffices for the proof of theorem (4.2) to establish a priori bounds C_1, C_2, C_3 and C_4 independent of λ such that

$$|x|_\infty \le C_1, |\dot{x}|_\infty \le C_2, |\ddot{x}|_\infty \le C_3, |\dddot{x}|_\infty \le C_4 \tag{1.53}$$

Verification of Equation (1.53)

Let $x(t)$ be a possible 2π −periodic solution of equation (1.45). The main tool to be used here in the verification is the function $w(x, y, z, u)$ defined by

$$w = \int_0^T {x^{(4)}}^2 dt + \int_0^T h_\lambda \ddot{x} x^{(4)} dt + \int_0^T a_4 \ddot{x}^2\, dt + \int_0^T \lambda p x^{(4)}\, dt \tag{1.54}$$

The time derivative $\dot{w}$ is

$$\dot{w} = {x^{(4)}}^2 + h_\lambda \ddot{x} x^{(4)} + a_4 \ddot{x}^2 - \lambda p x^4 \tag{1.55}$$

Integrating (1.55) with respect to (1.43) from $t = 0$ to $t = 2\pi$ using (1.44) we obtain

$$\int_0^{2\pi} {x^{(4)}}^2 dt + \int_0^{2\pi} h_\lambda \ddot{x} x^{(4)} dt + \int_0^{2\pi} a_4 \ddot{x}^2\, dt \le \int_0^{2\pi} |\lambda||p| x^{(4)}\, dt$$

Further simplification gives

$$\int_0^2 (x^4 + \frac{1}{2} h_\lambda \ddot{x})^2 dt + \int_0^{2\pi} (a_4 + \frac{1}{4} h_\lambda^2) \ddot{x}^2 dt \le C_1 \int_0^{2\pi} |x^4| \, dt \qquad (1.56)$$

Where we have used the boundedness of p and fact that $0 \le \lambda \le 1$ to achieve (1.56)

In particular,

$$\int_0^2 (x^4 + \frac{1}{2} h_\lambda \ddot{x})^2 dt \le C_1 \int_0^{2\pi} |x^4| \, dt \qquad (1.57)$$

In pursuit of proof of (1.29), we need the following lemma in Ezeilo and Onyia (1984)

Lemma 4.3 Let $x = x(t)$ be a twice differentiable 2π −periodic function of t. Then

$$\int_0^{2\pi} (\ddot{x} + vx)^2 dt \ge \beta^2 \int_0^{2\pi} x^2 dt \qquad (1.58)$$

Lemma 4.4 Let $x = x(t)$ be a twice continuously differentiable 2π −periodic function of t. Then there are constants C_2, C_3 such that

$$\int_0^{2\pi} \dot{x}^2 \, dt \le C_2 \int_0^{2\pi} (\ddot{x} + vx)^2 \, dt \qquad (1.59)$$

and

$$\int_0^{2\pi} \ddot{x}^2 \, dt \le C_3 \int_0^{2\pi} (\ddot{x} + vx)^2 \, dt \qquad (1.60)$$

Since $y = \dot{x},\ z = \ddot{x}$ without loss of generality

$$\int_0^{2\pi} (x^{(4)} + \alpha \ddot{x})^2 \, dt \le C_1 \int_0^{2\pi} |x^4| dt \qquad (1.61)$$

is identical to

$$\int_0^{2\pi} (\ddot{z} + \alpha z)^2 \, dt \le C_1 \int_0^{2\pi} |\ddot{z}| \, dt \qquad (1.62)$$

Therefore

$$\int_0^{2\pi} (\ddot{z} + \alpha z)^2 \ge \beta^2 \int_0^{2\pi} z^2 \, dt \qquad (1.63)$$

which is analogous from Lemma (4.1) of (1.58). Also

$$\int_0^{2\pi} \ddot{z}^2 \, dt \le C_3 \int_0^{2\pi} (\ddot{z} + \alpha z)^2 dt \qquad (1.64)$$

From (1.61) and (1.64) we have

$$\int_0^{2\pi} (x^4)^2 (t) \, dt \le C_3 C_1 \int_0^{2\pi} \left| x^{(4)} \right| dt \le C_3 C_1 (2\pi)^{\frac{1}{2}} |x^4|_2 \qquad (1.65)$$

by Schwartz's inequality. Thus

$$\left| x^{(4)} \right| \le C_4 \qquad (1.66)$$

Where $C_4 = C_3 C_1 (2\pi)^{\frac{1}{2}}$

From (4.39) and because of (1.55) with $r = 3$

$$|\dddot{x}|_\infty \leq C_5 \quad (1.67)$$

Since $\dot{x}(0) = \dot{x}(2\pi)$, there exist $\ddot{x}(\tau_1) = 0$ for some $(\tau_1) \in [0,2\pi]$ such that the identity

$\ddot{x}(t) = \ddot{x}(\tau_1) + \int_{\tau_1}^{t} \dddot{x}(s)ds$ holds.

Therefore $\max_{0 \leq t \leq 2\pi} |\ddot{x}(t)| \leq \int_0^{2\pi} |\dddot{x}(t)|\, dt \leq (2\pi)^{\frac{1}{2}} |\dddot{x}|_2$ by Schwartz's inequality. In view of (1.67)

$$|\ddot{x}|_\infty \leq C_6 \quad (1.68)$$

Again $x(0) = x(2\pi)$ implies that there exists $\dot{x}(\tau_2) = 0$ for some $\tau_2 \in [0,2\pi]$ such that the identity $\dot{x}(t) = \dot{x}(\tau_2) + \int_{\tau_1}^{t} \ddot{x}(s)ds$ holds.

From which $\max_{0 \leq t \leq 2\pi} |\dot{x}(t)| \leq \int_0^{2\pi} |\ddot{x}|\, dt \leq (2\pi)^{\frac{1}{2}} |\ddot{x}|_2$ by Schwartz's inequality. From (1.68) we have

$$|\dot{x}|_\infty \leq C_7 \quad (1.69)$$

It remain only the first inequality in (1.53) and for theorem 4.1. Now integrating (1.45) with respect to t from $t = 0$ to $t = 2\pi$ and using (1.44) we have

$$\int_0^{2\pi} a_4 x dt = \int_0^{2\pi} \lambda p dt - \int_0^{2\pi} \lambda g(\dddot{x}) dt - \int_0^{2\pi} h_\lambda(\ddot{x}) dt - \int_0^{2\pi} a_3 \dot{x} dt \quad (1.70)$$

with bounds on $\dddot{x}, \ddot{x}$ and $\dot{x}$ established in (1.65), (1.66) and (1.67) respectively and the boundedness of p together with the fact that $0 \leq \lambda \leq 1$, the expression on the right hand side of equation (1.68) is bounded. That is

$$\int_0^{2\pi} \lambda p dt - \int_0^{2\pi} \lambda g(\dddot{x}) dt - \int_0^{2\pi} h_\lambda(\ddot{x}) dt - \int_0^{2\pi} a_3 \dot{x} dt \leq C_8$$

Therefore

$$\int_0^{2\pi} |a_4 x|\, dt \leq C_8 \quad (1.71)$$

Further simplification yields

$$\int_0^{2\pi} |x|\, dt \leq C_9 \quad (1.72)$$

Where $C_9 = a_4^{-1} C_8;\ a_4 \neq 0$. Therefore $\max_{0 \leq t \leq 2\pi} |x(t)| \leq C_9$. Hence

$$|x|_\infty \leq C_{10} \quad (1.73)$$

Using the estimate of (1.67), (1.68), (1.69) and (1.73) the proof of theorem 4.1 is established.

5. Conclusions

In the present results for fourth order differential equation, integrated equations are used in estimating a priori bounds which has the following advantages

(a) It yields itself estimates for $\int_0^{2\pi} x^2(t)\ddot{x}\,dt$ or $\int_0^{\pi} x^2\dot{x}\,dt$

(b) The integral taken between 0 to 2π leads to vanishing of some terms due to 2π −periodicity condition.

(c) Integrated equations are easier to construct and are rare in literature unlike the Lyapunov function which are complex and cumbersome to construct for higher order nonlinear differential equations.

(d) The theorems have been develop through a progression from constant coefficient equations and the nonlinear equation. Also other relevant conditions were added to the hypotheses of the theorems.

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